TOPICS ON THE GEOMETRY AND CLASSIFICATION OF BANACH LATTICES

BY

MARY ANGELICA GRAMCKO-TURSI

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois Urbana-Champaign, 2021

Urbana, Illinois

Doctoral Committee:

Professor Marius Junge, Chair
Research Professor Timur Oikhberg, Director of Research
Professor Denka Kutzarova
Assistant Professor Anush Tserunyan

Abstract

We examine topics related to the geometric structure of Banach lattices of various classes, their properties, and classification using tools from functional analysis and mathematical logic. This work can be roughly divided into four parts. The first part (Chapter 2) presents several geometric results on Banach lattice analogues of classical Banach space theorems which ground results from later sections. The second part (Chapters 3 and 4) presents various results on the descriptive complexity of classes of Banach lattices and determines the complexity of the lattice isomorphism and isometry equivalence relations. The focus of the third part (Chapter 5) is the construction of a lattice isometrically universal separable "Gurarij" Banach lattice by combining properties of the geometric structure of Banach lattices with Fraïssé machinery that was developed in the context of continuous logic. Finally, we return to geometric considerations in the fourth and last part in chapter 6, which describes a method of renorming AM spaces so that the only lattice isometry is the trivial isometry.

Acknowledgments

Many thanks, first of all, to Timur Oikhberg for being an excellent advisor during my time at the University of Illinois. Through his guidance and knowledge, I was given the conditions to grow from a more passive learner of mathematics to an active researcher, and his suggestions on certain paths to take bore much fruit. But most important, through our meetings, he exemplified and imparted to me a proper attitude of genuine curiosity, patience, and humility, all of which are necessary to overcome various mathematical obstacles while appreciating the beauty in the objects that we studied.

I also thank Marius Junge, Denka Kutzarova, and Anush Tseryunan for their guidance and insights in my defense, through the various conversations we had on topics related to analysis and logic, and in the classes that I took. These interactions all helped me to gain an appreciation for various frameworks in analysis and logic as well as strengthening my own mathematical intuition.

I am indebted to my friends Mina Nahvi, Hanna Kim, Souktik Roy, and Robert Dicks for being such great support for me by caring for Raphael. It means a lot to me as a mathematician and as a mother. Thanks also to Jesse Huang and Minh Tran for the camaraderie and deep conversations we had in my office as well as for being examples to me of a proper ethos in dedication to mathematical research.

I also thank my undergraduate math professors Dr. John Coleman, Dr. David Burton, and Dr. Mary Salter from Franciscan University of Steubenville for their many classes with excellent teaching, illuminating conversations about mathematical topics, and their guidance and encouragement.

I am especially indebted to my husband, Dane, without whose constant support this work would have been impossible, especially during moments of increased stress and worries. On one end, you sacrificed much for my sake and put up with me all these years, and on the other end, you have helped me to grow as a wife, mother, mathematician, and person as a whole. Thanks also to Raphael: your coming into this world gave both my life and my work a new purpose. And to the little one kicking inside me, thanks for keeping me company in these final months in Illinois. Thank you also to my parents Artemis and Claudio for creating an environment where I could flourish and for encouraging me to pursue excellence, as well as my siblings Paolo and Jessie for accepting my eccentricities and, from time to time, being willing to listen to my bumbling attempts to explain higher level math in accessible ways.

Finally, the greatest gratitude is owed to God who has provided me with the opportunities and support necessary to accomplish this work and directed my steps at crucial points to lead me where I am today. May this work, which explores but a small part of the mathematical world of order and structure, be received as a prayer in the practice of my trade to Him who in His Wisdom sweetly orders all things.

Table of Contents

Chapter 1

Introduction

1.1 Motivation and overview

This thesis treats on various topics tied to the geometry, structure, and categorization of Banach lattices using tools both from analysis and mathematical logic. It includes results from three papers [64, 65, 70] as well as additional previously unpublished work. While the topics at a first glance seem unrelated, my hope is that it gives an adequate initial display of the interplay between the geometric and "definable" properties of Banach lattices.

Banach lattices are Banach spaces which, intuitively, carry with them our basic notions about order applied to functions. For instance, it is true that we do not consider $f(x) = x$ along [-1,1] to be "greater than" or "less than" $f(x) = 0$. Presumably though, we want to be able to say that if $f(x) > 0$ for all x in the specified domain, then we can say it is "greater" than 0. At least, then, we have a partial order which expands our notion of the ordering of $\mathbb{R}$. In addition, we want the norm to be related to the order: specifically, if $f \leq g$ (that is, if $0 \leq g - f$), and $0 \leq f$, then $\|f\| \leq \|g\|$. We also want the linear operations to interact well with this partial order in a manner similar to how order is preserved by multiplication by positive scalars or by addition. Finally, lattices are equipped with greatest lower and least upper bound operations. When applied to $\mathbb{R}$ these operations are quite simple: between two numbers, pick the smallest (or largest) one, or for real-valued functions f and g, one can pick pointwise minimum (or maximum) values to get $f \wedge g$ (or $f \vee g$).

The extra partial order, least upper bound, and greatest lower bound operations in Banach lattices is not too complicated. Many classical Banach spaces (ℓ_p, $L_p(\mu)$, $C(K)$) also come naturally equipped with the Banach lattice structure. Yet this additional structure results in some beautiful theorems that tell us much more about the geometry of a given space. It allows us isolate various properties that can completely characterize a class of lattices, often in a way that cannot be done for Banach spaces. In particular, we focus on two sets of tools from logic which make advantageous use of the lattice structure.

The first is that of descriptive set theory. Descriptive set theory techniques are used to answer certain questions about classes of structures according to notions of complexity. For example, a class not being Borel means that it is too complicated to be explicitly constructed in a "nice" way. Descriptive set theory also provides a way of describing how the resolution of a "problem" might be reduced to solving another "problem." This is captured in the notion of complexity in equivalence relations, and Borel reduction describes a concrete way of describing when equivalence (for example, if two spaces are appropriately isomorphic, or if they are in the same orbit of some group action) can be determined by another problem. Descriptive set theory tools have been applied for various structures in functional analysis (see, for instance [6] for a treatment on the interplay of descriptive set theory and Banach spaces).

The other set of tools we will be employing arise from continuous logic. Continuous logic expands on the basic notions found in model theory to metric spaces. Sentences over a metric structure are assigned some real value (typically between 0 and 1) rather than being true or false, while relations and functions, as well as formulas in general, are uniformly continuous functions. Under some mild conditions, one can derive continuous logic equivalents of the standard theorems in Model theory (see [14] for a fuller treatment). Banach lattices are axiomatizable in continuous logic, which means that we can apply its tools to understand them as models fulfilling a certain set of axioms.

The rest of Chapter 1 includes some definitions which feature in multiple chapters. After this, we prove some preliminary results underlying the rest of chapters, as well as several propositions regarding the relationships between lattices and certain characteristic objects. In particular we prove that several functions mapping a lattice, or a lattice and a point, to some object, like for example, another element or some closed subset, are Borel. These functions are used throughout the rest of the work in proving more advanced results.

Chapter 2, based on [64], presents various results on lattice analogues of geometric notions in Banach space theory like extreme points and convex hulls. We define the lattice versions: order extreme points and solid convex hulls, and prove various lattice versions of classical geometric theorems, including separation by positive functionals, "solid" Krein-Milman and Milman theorems, and a proof that the RNP is equivalent to the Solid Krein-Milman property.

Chapter 3 consists of results involving the descriptive complexity of various classes of Banach lattices. We show the Borelness of p-convex/concave lattices, order continuity, and subclasses of atomic lattices. With respect to the latter, the Fatou property comes into play, since it enables one to define band-like behavior in a Borel way. We then delve into classes of atomic order continuous rearrangement invariant (r.i) lattices. Our last section discusses the isomorphically universal separable order continuous Pelczynski lattice $\mathcal{V}$, which we use to determine the descriptive complexity of certain non-Borel classes of lattices such as KB lattices, dual lattices, and reflexive lattices.

Chapter 4 focuses on equivalence relations in Banach lattices. In particular, we show that the isomorphism equivalence relation of Banach lattices is analytically complete, and that the isometry relation is equivalent to the universal relation of group actions on standard Borel spaces.

Chapter 5 deals with the similar concepts of homogeneity (coming from continuous Logic) and universal disposition (coming from Banach space theory). Here, we show that finitely generated Banach lattices form a Fraïssé class, and thus there exists an isometrically universal, approximately ultra-homogeneous separable Banach lattice. We also give various constructions of this lattice that reveal some of its interesting properties. This chapter combines tools and concepts from continuous Logic as well as ideas in functional analysis. We end with the construction of a Pelczynski lattice with homogeneity properties using techniques established in the chapter.

Chapter 6 presents some results on equivalent renormings of lattices. Here we explore how separable AM-spaces can be renormed with equivalent renormings so that the only lattice isometry on the space is the identity. This chapter forms the beginning of some further investigations into the displayability of groups in Banach lattices which are still in preparation.

1.2 Basic definitions

The following terms are used throughout this work:

A **Banach lattice** is a Banach space equipped with a partial order $\leq$ and operations $\vee$ and $\wedge$, corresponding to least upper bound and greatest lower bound with respect to $\leq$ satisfying the following:

- For all $x, y, z \in X$, and $a > 0$, $x \leq y$ implies $x + z \leq y + z$ and $ax \leq ay$.
- Let $|x| = x \vee -x$. Then for all $x, y \in X$, $|x| \leq |y| \implies \|x\| \leq \|y\|$.

For references related to Banach lattices and their properties, see [61] and [3]. A map $\phi : X \to Y$ between Banach lattices is called a **lattice homomorphism** if it is a bounded linear map that preserves that lattice operations: that is, for all $x, y \in X$, we have $\phi(x) \wedge \phi(y) = \phi(x \wedge y)$. If $\|x\| = \|\phi(x)\|$ for all $x \in X$, then ϕ is a **lattice isometric embedding**. If ϕ is a bijection, we call it a **lattice isomorphism**, and if it preserves norms, we call it a **lattice isometry**. To check whether a linear map is also a lattice homomorphism, by [1, Theorem 1.34], it is enough to check that it is positive ($x \geq 0 \implies \phi(x) \geq 0$) and preserves disjointness. That is, if $x, y \in X$ and $x \perp y$ (i.e., $|x| \wedge |y| = 0$), then $\phi(x) \perp \phi(y)$. Sometimes, we might wish to focus on the amount of distortion in an isomorphism. Suppose $\phi : X \to Y$ is a lattice homomorphism such that there is some fixed M such that

$$1/M\|x\| < \|\phi(x)\| < M\|x\|$$

for all $x \in X$. We then call ϕ an **M-embedding**. If ϕ is an M-embedding for some M, we call it an **isomorphic embedding**, If ϕ is also a lattice isomorphism, we can call it an **M-isometry**.

Let X be a Banach Lattice and suppose $Y \subseteq X$ is a Banach sublattice. We say that Y is an **ideal** if for all $y \in Y$ and $z \in X$ such that $|z| \leq y$, $z \in Y$. By ideal, we mean specifically a closed ideal, unless stated otherwise. If $A \subseteq X$, call the smallest ideal containing all the elements in A the **ideal generated by A**. Similarly, if $x \in X$, we call the smallest Banach lattice ideal containing x the **principal ideal** generated by x. An ideal $Y \subseteq X$ is called a **band** if for all increasing nets $x_\alpha \uparrow x \in X$, with $0 \leq x_\alpha \in Y$, it follows that $x \in Y$. Y is called a **projection band** it is a band and there exists another band Z such that X is lattice isometric to $Y \oplus Z$. Note here that Y induces a projection: for any $x \in X$, you have $x = y + z$, with $y \in Y, z \in Z$, and y is disjoint from z. the map sending x to y is a lattice projection.

An element $e \in X$ is called an **atom** if $e > 0$ and for all $0 \leq x \leq e$, $x = re$ for some $r \in \mathbb{R}$. We say that a lattice is **atomic** if it is equal to the band generated by it atoms, and it is **atomless** if it has no atoms. We also say that an element $x \in X_+$ is atomless if the ideal generated by x is atomless. An element $x \in X_+$ is a **quasi-interior point** if for all $y \in X_+$, $y = \lim_n y \wedge (nx)$. A weaker notion is that of weak (order) units.

An element $x \in X_+$ is a weak unit of for all $y \in X_+$, $y = \sup_n y \wedge nx$. The difference between the two is that with quasi-interior points, there is a convergence in norm, while with weak units, it is only an order convergence. All quasi-interior points are weak units, but the opposite is not the case. For properties of quasi-interior points and weak units, see [3, pp. 266-267]).

We say that $A \subset X$ is **solid** if for $x \in X$ and $z \in C$, if $|x| \leq |z|$, then $x \in C$. In particular, $x \in X$ belongs to C if and only if $|x|$ does. Note that any solid set is automatically **balanced**; that is, $C = -C$. If we focus on X_+, we also say that $C \subset X_+$ is **positive-solid** if for any $x \in X_+$, the existence of $z \in C$ satisfying $x \leq z$ implies the inclusion $x \in C$.

A Banach Lattice X has a **Fatou norm** if for all sequences $x_n \uparrow x$, where $x_n, x \in X_+$, we also have $\|x_n\| \uparrow \|x\|$. We also say that X has a **weak Fatou norm** is there exists some constant M such that for all sequences of positive elements $0 \leq x_n \uparrow x$, we have $\sup \|x_n\| \geq M\|x\|$.

X is **order complete** if every order bounded subset of X has a least upper bound. Similarly X is σ-order complete if every sequence bounded above has an upper bound. Since we are mainly dealing with separable spaces, if X is σ-order complete, then it is also just order complete. We also say X is (σ)-**order continuous** if for all sequences (x_n) such that $x_n \downarrow 0$, we also have $\|x_n\| \to 0$. An equivalent definition is that any order bounded increasing sequence $0 \leq x_n$ converges in norm. X is **KB** if any norm bounded increasing sequence $0 \leq x_n$ converges in norm. Observe that all KB lattices are order continuous, but the converse is false (see for example, c_0).

The dominant structure in descriptive set theory is that of **Polish spaces**, that is, separable complete metrizable spaces. Separable Banach lattices whose metric induced by the lattice norm. In addition, the group of lattice isometries is a Polish space. Its topology is generated by the metric

$$d(g, h) = \sum 2^{-i}\|gx_i - hx_i\|$$

where $(x_i)_i$ is dense in the unit ball of X. This metric is not necessarily complete on X, but the metric $D(g, h) = d(g, h) + d(g^{-1}, h^{-1})$ is complete and generates the same topology.

Sometimes, we can "forget" the Polish topology on X and focus on the Borel σ-algebra $\beta(X)$. We call a measure space (X, μ) with $\mu \subseteq P(X)$ a σ-algebra, a **standard Borel space** if there exists a Polish metric on X such that μ is the Borel σ-algebra generated by the open sets from the metric. In this instance, the metric itself little importance, but the Borel structure formalizes the notion of of a set or class being topologically "definable" by countable set operations. Clearly Polish spaces are standard Borel spaces. Furthermore, any Borel subset of a Polish space is also standard Borel space (see [46, Theorem 13.1]). One can also show that various classes of objects also admit a standard Borel structure. For example, given a Polish space X, the **Effros-Borel space** $F(X)$ of closed subsets of X with a sigma algebra generated by sets of the form

$$\{F \subseteq X : F \cap U \neq \emptyset\}$$

with U open.

A subset $A \subseteq X$ with X a standard Borel space is **analytic** if there is standard Borel space Y and a Borel function $f : Y \to X$ such that $f(Y) = A$. In addition, we say that $X \backslash A$ is **co-analytic**. For a given space X, we let $\Sigma_1^1(X)$ be the set of analytic subsets of X, and $\Pi_1^1(X)$ be the co-analytic subsets of X. A theorem by Souslin states that for standard Borel spaces X, $\beta(X) = \Sigma_1^1(X) \cap \Pi_1^1(X)$ (see [46, Chapter 14] for a proof).

More generally, for a class Γ of sets in Polish (standard Borel) spaces, we call $\check{\Gamma}$ the class of the compliments of sets of Γ. For example, if $\Gamma = \Sigma_1^1$, $\check{\Gamma} = \Pi_1^1$. The following is taken from [46]: Let Γ be a class of sets in Standard Borel spaces. If Y is a standard Borel space, we call $A \subseteq Y$ **$\boldsymbol{\Gamma}$-hard** if for any standard Borel X and $B \in \Gamma(X)$, there exists a Borel function $f : X \to Y$ such that $B = f^{-1}(A)$. Furthermore, if $A \in \Gamma(Y)$, the we say that A is **$\boldsymbol{\Gamma}$-complete**.

1.3 Preliminaries: basics on Borel classes and maps

Our first task is to construct a suitable ambient space (specifically, a standard Borel space) whose elements are all the separable lattices. We then show that certain basic relations involving lattices, elements, or closed sets are also Borel, which will allow us to define various classes and other equivalence relations within a certain level in the projective hierarchy.

Suppose X is a separable Banach lattice. Then we can use the Kuratowski-Ryll-Nardsewski Theorem to get Borel functions $\psi_n^k : F(X^k) \to X^k$ with $F(X_k)$ the standard Effros-Borel space on X^k, where for all $k \in \mathbb{N}$ and $F \in F(X^k)$, $\overline{\{\psi_n^k(F) : \ n \in \mathbb{N}\}} = F$. From this we can get a dense enumeration of the Banach Lattice X and of each of its sublattices. From now on, for $k = 1$, we will let $\psi_n^1 = \psi_n$.

Proposition 1.3.1. *The sets $BL(X)$ of infinite dimensional Banach sublattices in $F(X)$ and $BL_f(X)$, of all Banach sublattices of X are Borel:*

Proof.

$$F \in BL(X) \iff \forall m, n \in \mathbb{N}, p, q \in \mathbb{Q}, \ (q\psi_m(F) + p\psi_n(F) \in F) \bigwedge \tag{1.1}$$

$$(\psi_n(F) \vee \psi_m(F) \in F) \bigwedge \forall k \exists n_1, ..., n_k \in \mathbb{N}^k \tag{1.2}$$

$$\neg \left[\forall M \in \mathbb{N} \exists q_1, ...q_k \left(\vee_i \, q_i = 1 \wedge \| \sum q_i \psi_{n_i}(F) \| < \frac{1}{M} \right) \right] \tag{1.3}$$

Here (1) gives linear closure, (2) gives closure under the lattice operations, and (3) gives the existence of arbitrarily many linearly independent vectors in F. To include all Banach lattices and thus define BL_f, simply remove condition (3). Since F is closed, it is sufficient to describe necessary conditions for a countable dense subset defined in the structure by the ψ_m's. Note that for separable spaces, the relation of inclusion $\{(x, F) \in B \times F(B) : x \in F\}$ is also Borel: Let U_n be a sequence of open sets, with $U_{n+1} \leq U_n$ such that $\mathrm{diam}(U_n) \to 0$, with $\{x\} = \bigcap_n U_n$. Then

$$x \in F \iff F \in \bigcap \{F : F \cap U_n \neq \emptyset\},$$

since x is a then a limit point and F is closed. □

A first attempt to classify up to isomorphism (or isometry) Banach lattices is to look at separable spaces with a certain number of atoms. We first note that the relation

$\mathcal{A}(y,F) \subseteq X \times BL(X)$ defined by

$$\mathcal{A}(y,F) \iff y \text{ is an atom of F and } \|y\| = 1$$

is G_δ. We can show this simply by defining it:

$$\begin{aligned}\mathcal{A}(y,F) \iff \|y\| = 1 \wedge y > 0 \bigwedge \\ \forall m, M \in \mathbb{N} \left(\exists q \in \mathbb{Q} \ \|y \wedge |\psi_m(F)| - qy\| < \frac{1}{M} \right)\end{aligned}$$

Also, note that for each F, the slice $\mathcal{A}_F$ of $\mathcal{A}$ is countable.For ease of notation, we use BL_f or BL to refer to the standard Borel space of all, or just infinite dimensional, separable Banach lattices in general. By [51], all separable Banach lattices isometrically embed into the space $\mathcal{U} := C(\Delta, L_1(0,1))$, so we can let $BL_f = BL_f(\mathcal{U})$. More generally, if $\mathcal{C}$ is a class of lattices, we can use the notation $\mathcal{C}(X)$ to refer to the sublattices of X that are in $\mathcal{C}$, but then also speak of $\mathcal{C}$ itself.

Theorem 1.3.2. *Let X be a separable Banach lattice. Let $\mathcal{A}_n \subseteq BL(X)$ denote the sublattices of infinite dimension with exactly n atoms, where $n = 0, ..., \omega$. Then each $\mathcal{A}_n$ is Borel.*

Proof. From the above we have that $\mathcal{A}$ is Borel. Note then that

$$F \in \mathcal{A}_n \iff \exists^n! \ y \ \mathcal{A}(y,F),$$

and $F \in \mathcal{A}_0 \iff \forall y \ \neg\mathcal{A}(y,F)$. By definition $\mathcal{A}_0$ is co-analytic, and from a result by Lusin (see [46, ch 19]), the set $\mathcal{A}_1$ is co-analytic. Using Lusin's result, it is a textbook problem in descriptive set theory to show that for all other n, the sets $\mathcal{A}_n$ are co-analytic. In addition, they are disjoint. Since X is separable, it can only contain countably many atoms with norm 1, so $BL(X) = \coprod_{n\in\omega+1} \mathcal{A}_n$. Co-analytic sets are closed under countable unions, so for each n, both $\mathcal{A}_n$ and its compliment $\bigcup_{i\neq n} \mathcal{A}_i$ are co-analytic, and thus both are analytic. By a corollary of Souslin's Theorem, if a set is both analytic and co-analytic, it is Borel. Hence each $\mathcal{A}_n$ is Borel. □

This does not exhaust the complexity of the isomorphism relation, but it does reveal that the number of sublattices up to isomorphism is at least the number of possible atoms that a sub lattice can have.

We also include some other results which are interesting in their own right and will be useful later on. They involve the Borelness of functions from $BL(X)$ (or $BL_f(X)$) to $F(X)$ that map Banach lattices to closed subsets of importance.

The maps $E \mapsto E_+$, $E \mapsto \mathbf{B}(E)$, $E \mapsto \mathbf{S}(E)$, $E \mapsto \mathbf{B}(E)_+$, and $E \mapsto \mathbf{S}(E)_+$ take a lattice to its positive cone, unit ball, unit sphere, positive unit ball, and positive unit sphere respectively. Note that in these definitions, it is often sufficient to quantify over a dense subset, since we are dealing with separable metrizable closed sets. Quantifying over countable sets is equivalent to taking countable unions or countable intersections. For the following, we let $SF(X)$ be the closed solid subsets of X.

Proposition 1.3.3. *Let X be a separable Banach lattice. Then the maps $\mathbf{B}, \mathbf{S} : BL(X) \to F(X)$ are Borel. In addition, the maps $_+ : BL(X) \to F(X)$, $\mathbf{B}_+ : BL(X) \to F(X)$, and $\mathbf{S}_+ : BL(X) \to F(X)$ are also Borel.*

Proof. By [46, Proposition 12.4], it is enough to show that the graphs are Borel. Consider the graph for $\mathbf{B}$, and let A be a sublattice of X. Then we have

$$\begin{aligned}\mathbf{B}(A) = B \iff \forall m \in \mathbb{N}, &\Big(\|\psi_m(B)\| \leq 1 \bigwedge \psi_m(B) \in A\Big) \bigwedge \\ \forall m \in \mathbb{N}, &\Big(\|\psi_m(A)\| \leq 1 \implies \psi_m(A) \in B\Big).\end{aligned}$$

Finally, consider the graphs for $\mathbf{S}$ and $\mathbf{S}_+$. Unlike the previous cases, it is possible for $\psi_m(A)$ to be disjoint from $\mathbf{S}(A)$, so we cannot simply write conventional conditions for being in $\mathbf{S}(A)$. Instead, we first define a clearly Borel map $D : X \to X$ by $x \mapsto \frac{x}{\|x\|}$ when $x \neq 0$ and $0 \mapsto 0$. From there, we define

$$\begin{aligned}\mathbf{S}(A) = B \iff \forall m \in \mathbb{N}, &\Big(\|\psi_m(B)\| = 1 \bigwedge \psi_m(B) \in A\Big) \bigwedge \\ \forall m \in \mathbb{N}, &\Big(\psi_m(A) \neq 0 \implies D(\psi_m(A)) \in B\Big).\end{aligned}$$

For each case, the graph is Borel, hence the functions are Borel. For $_+$, $\mathbf{B}_+$, and $\mathbf{S}_+$, the key fact is that if $F \in BL(X)$, $F = \mathbf{B}(E)$, or $F = \mathbf{S}(E)$, we have that $F_+ = \overline{|F|} = |F|$. Since $|\cdot|$ is continuous on X, by [46, Exercise 12.11ii], $\overline{|\cdot|}$ is Borel on $F(X)$, so in particular, $_+$ is a Borel function and $\mathbf{B}_+$ and $\mathbf{S}_+$ are compositions of Borel functions and thus Borel.

$\square$

We also consider maps from elements to sets or even lattices. For example, suppose we want not the unit ball, but the ball around some $x \in A$, where A is a sublattice. The following theorems also gives some Borel maps:

Proposition 1.3.4. *Let $x \in E \subseteq X$. Then the following partial functions are Borel:*

1. $(x, E, \delta) \mapsto \mathbf{B}_E(\delta, x)$, *mapping $x \in E$ to the ball of radius $\delta > 0$ in E,*

2. $(x, E, \delta) \mapsto \mathbf{S}_E(\delta, x)$, *mapping $x \in E$ to sphere of radius δ in E,*

3. $(x, E) \mapsto \mathbf{C}_E(x)$, *mapping $x \in E$ to the set $\{y \in E : y \geq x\}$,*

4. $(x, E) \mapsto I_E(x)$ *mapping $x \in E$ to the principal ideal generated by x in E.*

Proof. Note that these are only partial functions since they are only defined for when $x \in E$. Since the relation of inclusion in a Banach lattice is Borel, they can be understood as functions from a standard Borel space.) To prove (1), we simply adjust the function in Proposition 1.3.3, and prove the Borelness of the associated graph:

$$\begin{aligned} B = \mathbf{B}_E(\delta, x) \iff & B \subseteq E \bigwedge \forall m \|\psi_m(B) - x\| \leq \delta \\ & \bigwedge \forall m \Big(\|\psi_m(E) - x\| \leq \delta \implies \psi_m(E) \in B \Big) \end{aligned}$$

For (2) the result is similar, using the map D that was defined in Proposition 1.3.3. In line 1.5, we make the necessary shift and distortion by δ.

$$S = \mathbf{S}_E(\delta, x) \iff S \subseteq E \bigwedge \forall m \|\psi_m(S) - x\| = \delta \tag{1.4}$$

$$\bigwedge \forall m \Big(\psi_m(E) \neq x \implies x + \delta D(\psi_m(E) - x) \in F \Big) \tag{1.5}$$

For (3), All that is needed is a shift to x:

$$\begin{aligned} C = C_E(x) \iff & x \in E \bigwedge C \subseteq E \bigwedge \forall m (\psi_m(E) \geq x) \\ & \forall m (x + |\psi_m(E)| \in C) \end{aligned}$$

To prove (4), we have the following graph:

$$I = I_x \iff I \in BL_f \bigwedge x \in E \bigwedge I \subseteq E \tag{1.6}$$

$$\bigwedge \forall k \forall m \exists N \left(\|\psi_m(E_+) - (\psi_m(E_+) \wedge N|x|)\| < \frac{1}{k} \right) \tag{1.7}$$

$$\bigwedge \forall m (\psi_m(E_+) \wedge |x| \in I) \tag{1.8}$$

Lines 1.6 and 1.7 ensure that I is a sublattice of B contained in I_x, and line 1.8 (in combination with the fact that I is a lattice, ensures that $I_x \subseteq I$.

$\square$

Chapter 2

Geometry of Banach lattices: order extreme points and solid convex hulls

2.1 Introduction

This chapter is based on the joint work with Timur Oikhberg found in [64]. Many of these results are lattice analogues of results in classical Banach space theory, but they also include some additional, geometrically inspired results.

Let X be a Banach lattice, and suppose that $A \subseteq X$. Then $a \in A$ is an **order extreme point** of A if for all $x, y \in A$ with $t \in (0,1)$ and $a \leq (1-t)x + ty$, we have $x = a = y$. Observe that if $x \geq a$, then $a \leq \frac{1}{2}(x+a)$, so if a is order extreme, then $x = a$. Throughout the chapter, when we say a set A is bounded, we mean that it is norm bounded rather than order bounded.

In Section 2.2, we introduce notation and concepts that will be used for the chapter and establish some basic facts about order extreme points and convex hulls. In particular, a connection between order extreme points and traditional extreme points is made, and we end the section with the construction of Borel maps from sets to various hulls that will be used beyond the chapter.

In Section 2.3, we prove some order analogues of the Hahn- Banach Separation Theorem.

Section 2.4 uses the results from Section 2.3 to prove an "solid" Krein-Milman Theorem, namely that solid convex sets are the solid convex hulls of their order extreme points (see Theorem 2.4.1). We also prove a solid versions of Milman's Theorem (see Theorem 2.4.3).

In Section 2.5, we present two main results relating to the number of order extreme points. The first is that the unit ball of any infinite dimensional reflexive lattice has uncountably many order extreme points. The second result is a preliminary result from [70] placed here because of its geometric emphasis: we look specifically at finite dimensional lattices and show that such lattices with finitely many order extreme points are precisely the sublattices of ℓ_∞^m (ℓ_1^m) spaces.

The chapter concludes with Section 2.6, in which we define a solid analogue of the Krein-Milman Property and show that it is equivalent to the Radon-Nikodým Property.

2.2 Preliminaries: basics on order extreme points and solid convex hulls

We denote the closed unit ball (sphere) of a Banach space X is denoted by $\mathbf{B}(X)$ (resp. $\mathbf{S}(X)$). If X is a Banach lattice, and $C \subset X$, write $C_+ = C \cap X_+$, where X_+ stands for the positive cone of X. We will denote the set of order extreme points of C by $\mathrm{OEP}(C)$; the set of "classical" extreme points is denoted by $\mathrm{EP}(C)$.

Remark 2.2.1. The set of all extreme points of a compact metrizable set is G_δ. The same can be said for the set of order extreme points of C, whenever C is a closed solid bounded subset of a separable reflexive Banach lattice X. Since X is separable, the weak topology is induced by a metric d. For each n let F_n be the set of all $x \in C$ for which there exist $x_1, x_2, \in C$ with $x \leq (x_1 + x_2)/2$, and $d(x_1, x_2) \geq 1/n$. By compactness, F_n is closed. Indeed, assume $x_i \to x$, with $x_i \leq (x_i^1 + x_i^2)/2$ with $d(x_i^1, x_i^2) \geq 1/n$. Since C is compact, we can assume that $x_i^1 \to x_i$ and $x_i^2 \to x^2$, with $x \leq (x^1 + x^2)/2$ and $d(x^1, x^2) \geq 1/n$. Now observe that $\cup_n F_n$ is the complement of the set of all order extreme points.

Note that every order extreme point is an extreme point in the usual sense, but the converse is not true: for instance, $\mathbf{1}_{(0,1)}$ is an extreme point of $\mathbf{B}(L_\infty(0,2))_+$, but not its order extreme point. However, a connection between "classical" and order extreme points exists:

Theorem 2.2.2. *Suppose C is a solid subset of a Banach lattice X. Then a is an extreme point of C if and only if $|a|$ is its order extreme point.*

The proof of Theorem 2.2.2 uses the notion of a quasi-unit. Recall [61, Definition 1.2.6] that for $e, v \in X_+$, v is a **quasi-unit** of e if $v \wedge (e - v) = 0$. This terminology is not universally accepted: the same objects can be referred to as **components** [3], or **fragments** [66].

Proof. Suppose $|a|$ is order extreme. Let $0 < t < 1$ be such that $a = tx + (1-t)y$. Then since C is solid and $|a| \leq t|x| + (1-t)|y|$, one has $|x| = |y| = |a|$. Thus the latter inequality is in fact equality. Thus $|a| + a = 2a_+ = 2tx_+ + 2(1-t)y_+$, so $a_+ = tx_+ + (1-t)y_+$. Similarly, $a_- = tx_- + (1-t)y_-$. It follows that $x_+ \perp y_-$ and $x_- \perp y_+$. Since $x_+ + x_- = |x| = |y| = y_+ + y_-$, we have that $x_+ = x_+ \wedge (y_+ + y_-) = x_+ \wedge y_+ + x_+ \wedge y_-$ (since y_+, y_- are disjoint). Now since $x_+ \perp y_-$, the latter is just $x_+ \wedge y_+$, hence $x_+ \leq y_+$. By similar argument one can show the opposite inequality to conclude that $x_+ = y_+$, and likewise $x_- = y_-$, so $x = y = a$.

Now suppose a is extreme. It is sufficient to show that $|a|$ is order extreme for C_+. Indeed, if $|a| \leq tx + (1-t)y$ (with $0 \leq t \leq 1$ and $x, y \in C$), then $|a| \leq t|x| + (1-t)|y|$. Since $|a|$ is an order extreme point of C_+, we conclude that $|x| = |y| = |a|$, so $|a| = tx + (1-t)y = t|x| + (1-t)|y|$. The latter implies that $x_- = y_- = 0$, hence $x = |x| = |a| = |y| = y$.

Therefore, suppose $|a| \leq tx + (1-t)y$ with $0 \leq t \leq 1$, and $x, y \in C_+$. First show that $|a|$ is a quasi-unit of x (and by similar argument of y). To this end, note that $a_+ - tx \wedge a_+ \leq (1-t)y \wedge a_+$. Since C is solid,

$$C \ni z_+ := \frac{1}{1-t}(a_+ - tx \wedge a_+)$$

and similarly, since $a_- - tx \wedge a_- \leq (1-t)y \wedge a_-$,

$$C \ni z_- := \frac{1}{1-t}(a_- - tx \wedge a_-)$$

These inequalities imply that $z_+ \perp z_-$, so they correspond to the positive and negative parts of some $z = z_+ - z_-$. Also, $z \in A$ since $|z| \leq |a|$. Now $a_+ = t(x \wedge \frac{a_+}{t}) + (1-t)z_+$ and $a_- = t(x \wedge \frac{a_-}{t}) + (1-t)z_+$. In addition, $|x \wedge \frac{a_+}{t} - x \wedge \frac{a_-}{t}| \leq x$, so since C is solid,

$$z' := x \wedge \frac{a_+}{t} - x \wedge \frac{a_-}{t} \in C.$$

Therefore $a = a_+ - a_- = tz' + (1-t)z$. Since a is an extreme point, $a = z$, hence

$$(1-t)z_+ = (1-t)a_+ = a_+ - tx \wedge a_+$$

so $tx \wedge a_+ = ta_+$ which implies that $(t(x - a_+)) \wedge ((1-t)a_+) = 0$. As $0 < t < 1$, we have that a_+ (and likewise a_-) is a quasi-unit of x (and similarly of y). Thus $|a|$ is a quasi-unit of x and of y.

Now let $s = x - |a|$. Then $a + s, a - s \in C$, since $|a \pm s| = x$. We have

$$a = \frac{a-s}{2} + \frac{a+s}{2},$$

but since a is extreme, s must be 0. Hence $x = |a|$, and similarly $y = |a|$. $\square$

The situation is different if C is a positive-solid set: the paragraph preceding Theorem 2.2.2 shows that C can have extreme points which are not order extreme. If, however, a positive-solid set satisfies certain compactness conditions, then some connections between extreme and order extreme points can be established; see Proposition 2.4.11, and the remark following it.

If C is a subset of a Banach lattice X, denote by $\mathrm{S}(C)$ the **solid hull** of C, which is the smallest solid set containing C. It is easy to see that $\mathrm{S}(C)$ is the set of all $z \in X$ for which there exists $x \in C$ satisfying $|z| \leq |x|$. Clearly $\mathrm{S}(C) = \mathrm{S}(|C|)$, where $|C| = \{|x| : x \in C\}$. Further, we denote by $\mathrm{CH}(C)$ the convex hull of C. For future reference, observe:

Proposition 2.2.3. *If X is a Banach lattice, then* $\mathrm{S}(\mathrm{CH}(|C|)) = \mathrm{CH}(\mathrm{S}(C))$ *for any* $C \subset X$.

Proof. Let $x \in \mathrm{CH}(\mathrm{S}(C))$. Then $x = \sum a_i y_i$, where $\sum a_i = 1, a_i > 0$, and $|y_i| \leq |k_i|$ for some $k_i \in C$. Then

$$|x| \leq \sum a_i |y_i| \leq \sum a_i |k_i| \in \mathrm{CH}(|C|),$$

so $x \in \mathrm{S}(\mathrm{CH}(|C|))$. If $x \in \mathrm{S}(\mathrm{CH}(|C|))$, then

$$|x| \leq \sum_1^n a_i y_i, \quad y_i \in |C|, \quad 0 < a_i, \quad \sum a_i = 1.$$

We use induction on n to prove that $x \in \mathrm{CH}(\mathrm{S}(C))$. If $n = 1$, $x \in \mathrm{S}(C)$ and we are done. Now, suppose we have shown that if $|x| \leq \sum_1^{n-1} a_i y_i$ then there are $z_1, ..., z_{n-1} \in \mathrm{S}(C)_+$ such that $|x| = \sum_1^{n-1} a_i z_i$. From there, we have that

$$|x| = (\sum_1^n a_i y_i) \wedge |x| \leq (\sum_1^{n-1} a_i y_i) \wedge |x| + (a_n y_n) \wedge |x|.$$

Now

$$0 \leq |x| - (\sum_1^{n-1} a_i y_i) \wedge |x| \leq a_n(y_n \wedge \frac{|x|}{a_n}).$$

Let $z_n := \frac{1}{a_n}(|x| - (\sum_1^{n-1} a_i y_i) \wedge |x|)$. By the above, $z_n \in \mathrm{S}(C)_+$. Furthermore,

$$\frac{1}{1-a_n}(|x| \wedge \sum_1^{n-1} a_i y_i) \leq \sum_1^{n-1} \frac{a_i}{1-a_n} y_i \in \mathrm{CH}(|C|),$$

so by induction there exist $z_1, .., z_{n-1} \in \mathrm{S}(C)_+$ such that

$$|x| \wedge (\sum_1^{n-1} a_i y_i) = \sum_1^{n-1} \frac{a_i}{1-a_n} z_i$$

Therefore $|x| = \sum_1^n a_i z_i$. Now for each n, $a_i z_i \leq |x|$, so $|x| = \sum \big((a_i z_i) \wedge |x|\big)$, and

$$a_i z_i = a_i z_i \wedge x_+ + a_i z_i \wedge x_- = a_i(z_i \wedge (\frac{x_+}{a_i}) + z_i \wedge (\frac{x_-}{a_i})).$$

Let $w_i = z_i \wedge (\frac{x_+}{a_i}) - z_i \wedge (\frac{x_-}{a_i})$. Note that $|w_i| = z_i$, so $w_i \in \mathrm{S}(C)$. It follows that $x = \sum a_i w_i \in \mathrm{CH}(\mathrm{S}(C))$. □

For $C \subset X$ (as before, X is a Banach lattice) we define the **solid convex hull** of C to be the smallest convex, solid set containing C, and denote it by $\mathrm{SCH}(C)$; the norm (equivalently, weak) closure of the latter set is denoted by $\mathrm{CSCH}(C)$, and referred to as the **closed solid convex hull** of C.

Corollary 2.2.4. *Let $C \subseteq X$. Then*

1. $\mathrm{SCH}(C) = \mathrm{CH}(\mathrm{S}(C)) = \mathrm{SCH}(|C|)$, *and consequently,* $\mathrm{CSCH}(C) = \mathrm{CSCH}(|C|)$.

2. *If $C \subseteq X_+$, then* $\mathrm{SCH}(C) = \mathrm{S}(\mathrm{CH}(C))$.

Proof. (1) Suppose $C \subseteq D$, where D is convex and solid. Then $\mathrm{CH}(\mathrm{S}(C)) \subseteq D$. Consequently, $\mathrm{CH}(\mathrm{S}(C)) \subset \mathrm{SCH}(C)$. On the other hand, by Proposition 2.2.3, $\mathrm{CH}(\mathrm{S}(C))$ is also solid, so $\mathrm{SCH}(C) \subseteq \mathrm{CH}(\mathrm{S}(C))$. Thus, $\mathrm{SCH}(C) = \mathrm{CH}(\mathrm{S}(C)) = \mathrm{CH}(\mathrm{S}(|C|)) = \mathrm{SCH}(|C|)$.
(2) This follows from (1) and the equality in Proposition 2.2.3. □

Remark 2.2.5. The order in which one takes solid or convex hulls is important, as it is not the case that for any $C \subseteq X$, we have $\mathrm{S}(\mathrm{CH}(C)) = \mathrm{SCH}(C)$. For example, if $C = \{(-2, 1), (-1, -2)\} \subseteq \mathbb{R}^2$. Then $\mathrm{S}(\mathrm{CH}(C))$ is not even convex.

Remark 2.2.6. The two examples below show that $\mathrm{S}(C)$ need not be closed, even if C itself is. Example (1) exhibits an unbounded closed set C with $\mathrm{S}(C)$ not closed; in example (2), C is closed and bounded, but the ambient Banach lattice needs to be infinite dimensional.

(1) Let X be a Banach lattice of dimension at least two, and consider disjoint norm one $e_1, e_2 \in \mathbf{B}(X)_+$. Let $C = \{x_n : n \in \mathbb{N}\}$, where $x_n = \frac{n}{n+1} e_1 + n e_2$. Now, C is norm-closed: if $m > n$, then $\|x_m - x_n\| \geq \|e_2\| = 1$. However, $\mathrm{S}(C)$ is not closed: it contains re_1 for any $r \in (0, 1)$, but not e_1.

(2) If X is infinite dimensional, then there exists a closed *bounded* $C \subset X_+$, for which $\mathrm{S}(C)$ is not closed. Indeed, find disjoint norm one elements $e_1, e_2, \ldots \in X_+$. For $n \in \mathbb{N}$ let $y_n = \sum_{k=1}^{n} 2^{-k} e_k$ and $x_n = y_n + e_n$. Then clearly $\|x_n\| \leq 2$ for any n; further, $\|x_n - x_m\| \geq 1$ for any $n \neq m$, hence $C = \{x_1, x_2, \ldots\}$ is closed. However, $y_n \in \mathrm{S}(C)$ for any n, and the sequence (y_n) converges to $\sum_{k=1}^{\infty} 2^{-k} e_k \notin \mathrm{S}(C)$.

However, under certain conditions we can show that the solid hull of a closed set is closed.

Proposition 2.2.7. *A Banach lattice X is reflexive if and only if, for any norm closed, bounded convex $C \subset X_+$, $\mathrm{S}(C)$ is norm closed.*

Proof. Support first X is reflexive, and C is a norm closed bounded convex subset of X_+. Suppose (x_n) is a sequence in $\mathrm{S}(C)$, which converges to some x in norm; show that x belongs to $\mathrm{S}(C)$ as well. Clearly $|x_n| \to |x|$ in norm. For each n find $y_n \in C$ so that $|x_n| \leq y_n$. By passing to a subsequence if necessary, we assume that the sequence (y_n) converges to some $y \in X$ in the weak topology. For convex sets, norm and weak closures coincide, hence y belongs to C. For each n, $\pm x_n \leq y_n$; passing to the weak limit gives $\pm x \leq y$, hence $|x| \leq y$.

Now suppose X is not reflexive. By [3, Theorem 4.71], there exists a sequence of disjoint elements $e_i \in \mathbf{S}(X)_+$, equivalent to the natural basis of either c_0 or ℓ_1.

First consider the c_0 case. Let C be the closed convex hull of

$$x_1 = \frac{e_1}{2},\ x_n = \big(1 - 2^{-n}\big) e_1 + \sum_{j=2}^{n} e_j \ (n \geq 2).$$

We shall show that any element of C can be written as $ce_1 + \sum_{i=2}^{\infty} c_i e_i$, with $c < 1$. This will imply that $\mathrm{S}(C)$ is not closed: clearly $e_1 \in \overline{\mathrm{S}(C)} \backslash \mathrm{S}(C)$.

The elements of $\mathrm{CH}(x_1, x_2, \ldots)$ are of the form $\sum_{i=1}^{\infty} t_i x_i = ce_1 + \sum_{i=2}^{\infty} c_i e_i$; here, $t_i \geq 0$, $t_i \neq 0$ for finitely many values of i only, and $\sum_i t_i = 1$. Note that $c_i = \sum_{j=i}^{\infty} t_i$ for $i \geq 2$ (so $c_i = 0$ eventually); for convenience, let $c_1 = \sum_{j=1}^{\infty} t_i = 1$. Then $t_i = c_i - c_{i+1}$; Abel's summation technique gives

$$c = \sum_{i=1}^{\infty} \left(1 - 2^{-i}\right) t_i = 1 - \sum_{i=1}^{\infty} 2^{-i} \left(c_i - c_{i+1}\right) = \frac{1}{2} + \sum_{j=2}^{\infty} 2^{-j} c_j.$$

Now consider $x \in C$. Then x is the norm limit of the sequence

$$x^{(m)} = c^{(m)} e_1 + \sum_{i=2}^{\infty} c_i^{(m)} e_i \in \mathrm{CH}(x_1, x_2, \ldots);$$

for each m, the sequence $(c_i^{(m)})$ has only finitely many non-zero terms, $c^{(m)} = \frac{1}{2} + \sum_{j=2}^{\infty} 2^{-j} c_j^{(m)}$, and for all $m, n \in \mathbb{N}$, $|c_i^{(m)} - c_i^{(n)}| \leq \|x^{(m)} - x^{(n)}\|$. Thus, $x = ce_1 + \sum_{i=2}^{\infty} c_i e_i$, with $c = \frac{1}{2} + \sum_{j=2}^{\infty} 2^{-j} c_j$. As $0 \leq c_j \leq 1$, and $\lim_j c_j = 0$, we conclude that $c < 1$, as claimed.

Now suppose (e_i) are equivalent to the natural basis of ℓ_1. Let C be the closed convex hull of the vectors

$$x_n = \left(1 - 2^{-n}\right) e_1 + e_n \ (n \geq 2),$$

and show that $e_1 \in \overline{\mathrm{S}(C)} \backslash \mathrm{S}(C)$. Note that

$$C = \Big\{ \Big(\sum_{i=2}^{\infty} \left(1 - 2^{-n}\right) t_i \Big) e_1 + \sum_{i=2}^{\infty} t_i e_i : t_2, t_3, \ldots \geq 0, \sum_{i=2}^{\infty} t_i = 1 \Big\}.$$

Clearly e_1 belongs to $\overline{\mathrm{S}(C)}$, but not to $\mathrm{S}(C)$. $\square$

We end this section with a connection to descriptive set theoretic tools with some applications. These are used to characterize certain kinds of classes later:

Lemma 2.2.8. *For all Banach sublattices X and $F \in F(X)$, the relation $F \in SF(E)$ is Borel a Borel subset of $BL(X) \times F(X)$.*

Proof. We first show that

$$F \in SF(E) \iff F \subseteq E \bigwedge \forall m,n \in \mathbb{N}$$
$$\big(\psi_m(E)_+ \wedge |\psi_n(F)|\big) - \big(\psi_m(E)_- \wedge |\psi_n(F)|\big) \in F.$$

Clearly the above is true if F is solid. For the opposite implication, let $x \in F$, and suppose $y \in E$ is such that $|y| \leq |x|$. The function $f(u,v) = u_+ \wedge |v| - (u_- \wedge |v|)$ is continuous, so given $\varepsilon > 0$, find $m,n \in \mathbb{N}$ with $\psi_m(E)$ sufficiently close to y and $\psi_n(F)$ sufficiently close to x such that $\|f(x,y) - f(\psi_m(E), \psi_m(y))\| < \varepsilon$. Now since $|y| \leq |x|$, we have $f(x,y) = y$, so $\|y - f(\psi_m(E), \psi_m(y))\|$ is arbitrarily close to an element in F. $\square$

The above implies that the solid subsets of X itself are Borel.

Proposition 2.2.9. *Let X be a separable lattice, E a sublattice of X, and F be a closed subset of E. Then the following partial functions are Borel:*

1. $F \mapsto \overline{\mathrm{CH}}(F)$*, mapping $F \in F(X)$ to its closed convex hull.*
2. $(F,E) \mapsto \bar{\mathrm{S}}_E(F)$ *mapping $F \subseteq E$ to its closed solid hull in E.*
3. $(F,E) \mapsto \bar{\mathrm{S}}_E(F)_+$ *mapping $F \subseteq E_+$ to its closed positive-solid hull in E.*
4. $(F,E) \mapsto \mathrm{CSCH}_E(F)$*, mapping F to its closed solid convex hull in E.*

Proof. We show the graphs are Borel:

For (1), let $m \in \mathbb{N}$, and consider the Borel maps $\psi_i^m : F(X^m) \to X^m$. Then we have:

$$C = \overline{\mathrm{CH}}(F) \iff \forall m \forall i \forall q \in \mathbb{Q}^m \Big(\sum q_j = 1 \implies \psi_i^m(F) \cdot q \in C \Big) \bigwedge \tag{2.1}$$
$$\forall m,k \in \mathbb{N}\ \exists n,l \in \mathbb{N}\ \exists q \in \mathbb{Q}_+^n \Big(\sum q_j = 1 \bigwedge \tag{2.2}$$
$$\|\psi_m(C) - \psi_l^n(F) \cdot q\| < \frac{1}{k} \Big) \tag{2.3}$$

Line 2.1 guarantees that C contains the closed convex hull of F, while the rest guarantees that C is contained in it. For (2), we use a similar argument:

$$G = \bar{\mathrm{S}}_E(F) \iff F \subseteq G \subseteq E \bigwedge G \in SF(E) \bigwedge \forall m, k\ \exists y, y' \tag{2.4}$$

$$\left(y \in F \bigwedge y' \in E \bigwedge |y'| \leq |y| \bigwedge \|\psi_m(G) - y'\| < \frac{1}{k} \right) \tag{2.5}$$

Line 2.4 guarantees that G is a E-solid set containing F (a Borel relation by Lemma 2.2.8), and line 2.5 guarantees that G is generated by $S(F)$. The function is analytic, so by [46, Theorem 14.12], it is Borel.

For (3), observe that $\bar{\mathrm{S}}_E(F)_+ = \mathrm{S}_E(F) \cap E_+ = |\bar{\mathrm{S}}_E(F)|$. The function $|\cdot|$ is continuous on X, so by [46, Exercise 12.11ii], $\overline{|\cdot|}$ is Borel on $F(X)$, (3) is a composition of Borel functions and is Borel.

For (4), we need only to note that $CSCH_E(F) = \overline{CH}(\bar{S}_E(F))$. By Proposition 2.2.4, we already have that $CSCH_E(F) = \overline{CH}(S_E(F)) \subseteq \overline{CH}(\bar{S}_E(F))$. However, $\bar{S}_E(F) \subseteq CSCH_B(F)$, the latter which is convex, hence taking the closed convex hull on both ends gives the opposite inequality. (4) is a composition of two Borel functions, hence it is Borel.

□

2.3 Separation by positive functionals

Throughout the section, X is a Banach lattice, equipped with a locally convex Hausdorff topology τ. This topology is called **sufficiently rich** if the following conditions are satisfied:

(i) The space X^τ of τ-continuous functionals on X is a Banach lattice (with lattice operations defined by Riesz-Kantorovich formulas).

(ii) X_+ is τ-closed.

Note that (i) and (ii) together imply that positive τ-continuous functionals separate points. That is, for every $x \in X \backslash \{0\}$ there exists $f \in X_+^\tau$ so that $f(x) \neq 0$. Indeed, without loss of generality, $x_+ \neq 0$. Then $-x_+ \notin X_+$, hence there exists $f \in X_+^\tau$ so that $f(x_+) > 0$. By [61, Proposition 1.4.13], there exists $g \in X_+^\tau$ so that $g(x_+) > f(x_+)/2$ and $g(x_-) < f(x_+)/2$. Then $g(x) > 0$.

Clearly, the norm and weak topologies are sufficiently rich; in this case, $X^\tau = X^*$. The weak* topology on X, induced by the predual Banach lattice X_*, is sufficiently rich as well; then $X^\tau = X_*$.

Proposition 2.3.1 (Separation)**.** *Suppose τ is a sufficiently rich topology on a Banach lattice X, and $A \subset X_+$ is a τ-closed positive-solid bounded subset of X_+. Suppose, furthermore, $x \in X_+$ does not belong to A. Then there exists $f \in X_+^\tau$ so that $f(x) > \sup_{a \in A} f(a)$.*

Lemma 2.3.2. *Suppose A and X are as above, and $f \in X^\tau$. Then $\sup_{a \in A} f(a) = \sup_{a \in A} f_+(a)$.*

Proof. Clearly $\sup_{a \in A} f(a) \leq \sup_{a \in A} f_+(a)$. To prove the reverse inequality, write $f = f_+ - f_-$, with $f_+ \wedge f_- = 0$. Fix $a \in A$; then

$$0 = \big[f_+ \wedge f_-\big](a) = \inf_{0 \leq x \leq a} \big(f_+(a - x) + f_-(x)\big).$$

For any $\varepsilon > 0$ we can find $x \in A$ so that $f_+(a - x), f_-(x) < \varepsilon$. Then $f_+(x) = f_+(a) - f_+(a - x) > f_+(a) - \varepsilon$, and therefore, $f(x) = f_+(x) - f_-(x) > f_+(a) - 2\varepsilon$. Now recall that $\varepsilon > 0$ and $a \in A$ are arbitrary. □

Proof of Proposition 2.3.1. Use Hahn-Banach Theorem to find f strictly separating x from A. By Lemma 2.3.2, f_+ achieves the separation as well. □

Remark 2.3.3. In this chapter, we do not consider separation results on general ordered spaces. Our reasoning will fail without lattice structure. For instance, Lemma 2.3.2 is false when X is not a lattice, but merely an ordered space. Indeed, consider $X = M_2$ (the space of real 2×2 matrices), $f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $A = \{ta_0 : 0 \leq t \leq 1\}$, where $a_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; one can check that $A = \{x \in M_2 : 0 \leq x \leq a_0\}$. Then $f|_A = 0$, while $\sup\limits_{x \in A} f_+(x) = 1$.

The reader interested in the separation results in the non-lattice ordered setting is referred to an interesting result of [38], recently re-proved in [4].

2.4 Solid convex hulls: theorems of Krein-Milman and Milman

Throughout this section, the topology τ is assumed to be sufficiently rich (defined in the beginning of Section 2.3).

Theorem 2.4.1 ("Solid" Krein-Milman)**.** *Any τ-compact positive-solid subset A of X_+ coincides with the τ-closed positive-solid convex hull of its order extreme points.*

Proof. Let A be a τ-compact positive-solid subset of X_+. Denote the τ-closed positive convex hull of $\mathrm{OEP}(A)$ by B; then clearly $B \subset A$. The proof of the reverse inclusion is similar to that of the "usual" Krein-Milman.

Suppose C is a τ-compact subset of X. We say that a non-void closed $F \subset C$ is an **order extreme subset** of C if, whenever $x \in F$ and $a_1, a_2 \in C$ satisfy $x \leq (a_1+a_2)/2$, then necessarily $a_1, a_2 \in F$. The set $\mathcal{F}(C)$ of order extreme subsets of C can be ordered by reverse inclusion (this makes C the minimal order extreme subset of itself). By compactness, each chain has an upper bound; therefore, by Zorn's Lemma, $\mathcal{F}(C)$ has a maximal element. We claim that these maximal elements are singletons, and they are the order extreme points of C.

We need to show that, if $F \in \mathcal{F}(C)$ is not a singleton, then there exists $G \subsetneq F$ which is also an order extreme set. To this end, find distinct $a_1, a_2 \in F$, and $f \in X_+^\tau$ which separates them – say $f(a_1) > f(a_2)$. Let $\alpha = \max_{x \in F} f(x)$, then $G = F \cap f^{-1}(\alpha)$ is a proper, order extreme subset of F.

Suppose, for the sake of contradiction, that there exists $x \in A \backslash B$. Use Proposition 2.3.1 to find $f \in X_+^\tau$ so that $f(x) > \max_{y \in B} f(y)$. Let $\alpha = \max_{x \in A} f(x)$, then $A \cap f^{-1}(\alpha)$ is an order extreme subset of A, disjoint from B. As noted above, this subset contains at least one extreme point. This yields a contradiction, as we started out assuming all order extreme points lie in B. $\square$

Corollary 2.4.2. *Any τ-compact solid subset of X coincides with the τ-closed solid convex hull of its order extreme points.*

Of course, there exist Banach lattices whose unit ball has no order extreme points at all – $L_1(0,1)$, for instance. However, an order analogue of [52, Lemma 1] holds.

Milman's theorem [67, 3.25] states that, if both K and $\overline{\mathrm{CH}(K)}^\tau$ are compact, then $\mathrm{EP}\big(\overline{\mathrm{CH}(K)}^\tau\big) \subset K$. An order analogue of Milman's theorem exists:

Theorem 2.4.3. *Suppose X is a Banach lattice.*

1. *If $K \subset X_+$ and $\overline{\mathrm{CH}(K)}^{\tau}$ are τ-compact, then $\mathrm{OEP}\big(\overline{\mathrm{SCH}(K)}^{\tau}\big) \subseteq K$.*
2. *If $K \subset X_+$ is weakly compact, then $\mathrm{OEP}(\mathrm{CSCH}(K)) \subseteq K$.*
3. *If $K \subset X$ is norm compact, then $\mathrm{OEP}(\mathrm{CSCH}(K)) \subseteq |K|$.*

The following lemma describes the solid hull of a τ-compact set.

Lemma 2.4.4. *Suppose a Banach lattice X is equipped with a sufficiently rich topology τ. If $C \subset X_+$ is τ-compact, then $\mathrm{S}(C)$ is τ-closed.*

Proof. Suppose a net $(y_i) \subset \mathrm{S}(C)$ τ-converges to $y \in X$. For each i find $x_i \in C$ so that $|y_i| \leq x_i$ – or equivalently, $y_i \leq x_i$ and $-y_i \leq x_i$. Passing to a subnet if necessary, we assume that $x_i \to x \in C$ in the topology τ. Then $\pm y \leq x$, which is equivalent to $|y| \leq x$. □

Proof of Theorem 2.4.3. (1) We first consider a τ-compact $K \subseteq X_+$. Milman's traditional theorem holds that $\mathrm{EP}\big(\overline{\mathrm{CH}(K)}^{\tau}\big) \subseteq K$. Every order extreme point of a set is extreme, hence the order extreme points of $\overline{\mathrm{CH}(K)}^{\tau}$ are in K. Therefore, by Lemma 2.4.4 and Corollary 2.2.4,

$$\overline{\mathrm{SCH}(K)}^{\tau} = \overline{\mathrm{S}(\mathrm{CH}(K))}^{\tau} \subseteq \mathrm{S}\big(\overline{\mathrm{CH}(K)}^{\tau}\big) = \{x : |x| \leq y \in \overline{\mathrm{CH}(K)}^{\tau}\}.$$

Thus, the points of $\overline{\mathrm{SCH}(K)}^{\tau} \backslash \overline{\mathrm{CH}(K)}^{\tau}$ cannot be order extreme due to being dominated by $\overline{\mathrm{CH}(K)}^{\tau}$. Therefore $\mathrm{OEP}\big(\overline{\mathrm{SCH}(K)}^{\tau}\big) \subseteq \mathrm{OEP}\big(\overline{\mathrm{CH}(K)}^{\tau}\big) \subseteq K$.

(2) Combine (1) with Krein's Theorem (see e.g. [31, Theorem 3.133]), which states that $\overline{\mathrm{CH}(K)}^{w} = \overline{\mathrm{CH}(K)}$ is weakly compact.

(3) Finally, suppose $K \subseteq X$ is norm compact. By Corollary 2.2.4, $\mathrm{CSCH}(K) = \mathrm{CSCH}(|K|)$. $|K|$ is norm compact, hence by [67, Theorem 3.20], so is $\overline{\mathrm{CH}(|K|)}$. By the proof of part (1), $\mathrm{OEP}(\mathrm{CSCH}(K)) \subseteq |K|$. □

We turn our attention to interchanging "solidification" and norm closure. We work with the norm topology, unless specified otherwise.

Lemma 2.4.5. *Let $C \subseteq X$, where X is a Banach lattice, and suppose that $\mathrm{S}(\overline{|C|})$ is closed. Then $\overline{\mathrm{S}(C)} = \mathrm{S}(\overline{|C|})$.*

Proof. One direction is easy: $\mathrm{S}(C) = \mathrm{S}(|C|) \subseteq \mathrm{S}(\overline{|C|})$, hence $\overline{\mathrm{S}(C)} \subseteq \overline{\mathrm{S}(\overline{|C|})} = \mathrm{S}(\overline{|C|})$.

Now consider $x \in \mathrm{S}(\overline{|C|})$. Then by definition, $|x| \leq y$ for some $y \in \overline{|C|}$. Take $y_n \in |C|$ such that $y_n \to y$. Then $|x| \wedge y_n \in \mathrm{S}(|C|) = \mathrm{S}(C)$ for all n. Furthermore,

$$|x_+ \wedge y_n - x_- \wedge y_n| = |x| \wedge y_n,$$

so, $x_+ \wedge y_n - x_- \wedge y_n \in \mathrm{S}(C)$. By norm continuity of $\wedge$,

$$x_+ \wedge y_n - x_- \wedge y_n \to x_+ \wedge y - x_- \wedge y = x,$$

hence $x \in \overline{\mathrm{S}(C)}$. □

Remark 2.4.6. The assumption of $\mathrm{S}(\overline{|C|})$ being closed is necessary: Remark 2.2.6 shows that, for a closed $C \subset X_+$, $\mathrm{S}(C)$ need not be closed.

Corollary 2.4.7. *Suppose $C \subseteq X$ is relatively compact in the norm topology. Then $\overline{\mathrm{S}(C)} = \mathrm{S}(\overline{C})$.*

Proof. The set $\overline{C}$ is compact, hence, by the continuity of $|\cdot|$, the same is true for $|\overline{C}|$. Consequently, $|\overline{C}| \subseteq \overline{|C|} \subseteq \overline{|\overline{C}|} = |\overline{C}|$, hence $|\overline{C}| = \overline{|C|}$. By Lemmas 2.4.4 and 2.4.5, $\mathrm{S}(\overline{C}) = \mathrm{S}(|\overline{C}|) = \mathrm{S}(\overline{|C|}) = \overline{\mathrm{S}(C)}$. □

Remark 2.4.8. In the weak topology, the equality $|\overline{C}| = \overline{|C|}$ may fail. Indeed, equip the Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$ with its uniform probability measure μ. Define $x_i \in L_2(\mu)$ by setting, for $t = (t_1, t_2, \ldots) \in \Delta$, $x_i(t) = t_i - 1/4$ (that is, x_i equals to either $3/4$ or $-1/4$, depending on whether t_i is 1 or 0). Then $C = \{x_i : i \in \mathbb{N}\}$ belongs to the unit ball of $L_2(\mu)$, hence it is relatively compact. It is clear that $\overline{C}$ contains $\mathbf{1}/4$ (here and below, $\mathbf{1}$ denotes the constant 1 function). On the other hand, $\overline{C}$ does not contain $\mathbf{1}/2$, which can be witnessed by applying the integration functional. Conversely, $\overline{|C|}$ contains $\mathbf{1}/2$, but not $\mathbf{1}/4$.

Remark 2.4.9. Relative weak compactness of solid hulls have been studied before. If X is a Banach lattice, then, by [3, Theorem 4.39], it is order continuous iff the solid hull of any weakly compact subset of X_+ is relatively weakly compact. Further, by [21], the following three statements are equivalent:

1. The solid hull of any relatively weakly compact set is relatively weakly compact.
2. If $C \subset X$ is relatively weakly compact, then so is $|C|$.

3. X is a direct sum of a KB-space and a purely atomic order continuous Banach lattice (a Banach lattice is called purely atomic if its atoms generate it, as a band).

Finally, we return to the connections between extreme points and order extreme points. As noted in the paragraph preceding Theorem 2.2.2, a non-zero extreme point of a positive-solid set need not be order extreme. However, we have:

Proposition 2.4.10. *Suppose τ is a sufficiently rich topology, and A is a positive-solid subset of X_+. Then if $x \in A$ is an extreme point, for any $0 < t < 1$, $x \leq ta + (1-t)b$ implies $x \leq a, b$. In particular, the set $\{y \in A : y \geq x\}$ is an order extreme subset of A.*

Proof. Suppose $x \leq ta + (1-t)b$, with $0 < t < 1$. Then $x \leq ta \wedge x + (1-t)b \wedge x$. Then $0 \leq x - t(a \wedge x/t) \leq (1-t)b \wedge x = (1-t)(b \wedge x/(1-t))$. Let $a' = a \wedge (x/t)$, and let $b' = \frac{x - ta'}{1-t}$. Then $x = ta' + (1-t)b'$, so $a' = x = b'$, so $x \leq a$. Similarly, $x = b \wedge ((x/(1-t))$, so $x \leq b$. □

Proposition 2.4.11. *Suppose τ is a sufficiently rich topology, and A is a τ-compact positive-solid convex subset of X_+. Then for any extreme point $a \in A$ there exists an order extreme point $b \in A$ so that $a \leq b$.*

Remark 2.4.12. The compactness assumption is essential. Consider, for instance, the closed set $A \subset C[-1, 1]$, consisting of all functions f so that $0 \leq f \leq \mathbf{1}$, and $f(x) \leq x$ for $x \geq 0$. Then $g(x) = x \vee 0$ is an extreme point of A; however, A has no order extreme points.

Proof. If a is not an order extreme point, then we can find distinct $x_1, x_2 \in A$ so that $2a \leq x_1 + x_2$. Then by Proposition 2.4.10, $a \leq x_1, x_2$. Now consider the τ-closed set $B = \{x \in A : x \geq a\}$. As in the proof of Theorem 2.4.1, we show that the family of τ-closed extreme subsets of B has a maximal element; moreover, such an element is a singleton $\{b\}$. It remains to prove that b is an order extreme point of A. Indeed, suppose $x_1, x_2 \in A$ satisfy $2b \leq x_1 + x_2$. A fortiori, $2a \leq x_1 + x_2$, hence, by the preceding paragraph, $x_1, x_2 \in B$. Thus, $x_1 = b = x_2$. □

2.5 On the number of order extreme points

It is shown in [53] that if a Banach space X is reflexive and infinite-dimensional Banach lattice, then $\mathbf{B}(X)$ has uncountably many extreme points. Here, we establish a similar lattice result.

Theorem 2.5.1. *If X is a reflexive infinite-dimensional Banach lattice, then $\mathbf{B}(X)$ has uncountably many order extreme points.*

Note that if X is a reflexive infinite-dimensional Banach lattice, then Theorems 2.2.2 and 2.5.1 imply that $\mathbf{B}(X)$ has uncountably many extreme points, re-proving the result of [53] in this case.

Proof. Suppose, for the sake of contradiction, that there were only countably many such points $\{x_n\}$. For each such x_n, we define $F_n = \{f \in \mathbf{B}(X^*)_+ : f(x_n) = \|f\|\}$. Clearly F_n is weak* (= weakly) compact.

By the reflexivity of X, any $f \in \mathbf{B}(X^*)$ attains its norm at some $x \in \mathrm{EP}(\mathbf{B}(X))$. Since $f(x) \leq |f|(|x|)$ we assume that any positive functional attains its norm at a positive extreme point in $\mathbf{B}(X)$. By Theorem 2.2.2, these are precisely the order extreme points. Therefore $\bigcup F_n = \mathbf{B}(X^*)_+$. By the Baire Category Theorem, one of these sets F_n must have non-empty interior in $\mathbf{B}(X^*)_+$.

Assume it is F_1. Pick $f_0 \in F_1$, and $y_1, ..., y_k \in X$, such that if $f \in \mathbf{B}(X^*)_+$ and for each y_i, $|f(y_i) - f_0(y_i)| < 1$, then $f \in F_1$. Without loss of generality, we assume that $\|f_0\| < 1$, and also that each $y_i \geq 0$.

Further, we can and do assume that there exist mutually disjoint $u_1, u_2, \ldots \in \mathbf{S}(X)_+$ which are disjoint from $y = \vee_i y_i$. Indeed, find mutually disjoint $z_1, z_2, \ldots \in \mathbf{S}(X)_+$. Denote the corresponding band projections by $P_1, P_2, \ldots$ (such projections exist, due to the σ-Dedekind completeness of X). Then the vectors $P_n y$ are mutually disjoint, and dominated by y. As X is reflexive, it must be order continuous, and therefore, $\lim_n \|P_n y\| = 0$. Find $n_1 < n_2 < \ldots$ so that $\sum_j \|P_{n_j} y\| < 1/2$. Let $w_i = \sum_j P_{n_j} y_i$ and $y_i' = 2(y_i - w_i)$. Then if $|(f_0 - g)(y_i')| < 1$, with $g \geq 0, \|g\| \leq 1$, it follows that

$$\begin{aligned} |(f_0 - g)(y_i)| &\leq \frac{1}{2}(|(f_0 - g)(y_i')| + |(f_0 - g)(w_i)|) \\ &\leq \frac{1}{2}(1 + \|f_0 - g\|\|w_i\|) < \frac{1}{2}(1 + 2 \cdot \frac{1}{2}) = 1 \end{aligned}$$

We can therefore replace y_i with y_i' to ensure sufficient conditions for being in F_1. Then the vectors $u_j = z_{n_j}$ have the desired properties. Let P be the band projection complementary to $\sum_j P_{n_j}$ (in other words, complementary to the the band projection of $\sum_j 2^{-j} u_j$); then $Py_i = y_i$ for any i.

By [61, Lemma 1.4.3 and its proof], there exist linear functionals $g_j \in \mathbf{S}(X^*)_+$ so that $g_j(u_j) = 1$, and $g_j = P_{n_j}^* g_j$. Consequently, the functionals g_j are mutually disjoint, and $g_j|_{\mathrm{ran}\, P} = 0$. For $j \in \mathbb{N}$ find $\alpha_j \in [1 - \|P^* f_0\|, 1]$ so that $\|f_j\| = 1$, where $f_j = P^* f_0 + \alpha_j g_j$. Then, for $1 \leq i \leq k$, $f_j(y_i) = (P^* f_0)(y_i) + \alpha_j g_j(y_i) = f_0(y_i)$, which implies that, for every j, f_j belongs to F_1, hence attains its norm at x_1.

On the other hand, note that $\lim_j g_j(x_1) = 0$. Indeed, otherwise, there exist $\gamma > 0$ and a sequence (j_k) so that $g_{j_k}(x_1) \geq \gamma$ for every k. For any finite sequence of positive numbers (β_k), we have

$$\sum_k |\beta_k| \geq \Big\| \sum_k \beta_k g_{j_k} \Big\| \geq \sum_k \beta_k g_{j_k}(x_1) \geq \gamma \sum_k |\beta_k|.$$

As the functionals g_{j_k} are mutually disjoint, the inequalities

$$\sum_k |\beta_k| \geq \Big\| \sum_k \beta_k g_{j_k} \Big\| \geq \gamma \sum_k |\beta_k|$$

hold for every finite sequence (β_k). We conclude that $\overline{\mathrm{span}}[g_{j_k} : k \in \mathbb{N}]$ is isomorphic to ℓ_1, which contradicts the reflexivity of X. Thus, $\lim_j g_j(x_1) = 0$, hence $\lim_j f_j(x_1) = f_0(Px_1) \leq \|f_0\| < 1$. $\square$

Corollary 2.5.2. *Suppose C is a closed, bounded, solid, convex subset of a reflexive Banach lattice, having non-empty interior. Then C contains uncountably many order extreme points.*

Proof. We assume without loss of generality that $\sup_{x \in C} \|x\| = 1$. Note that 0 is an interior point of C. Indeed, suppose x is an interior point. Pick $\varepsilon > 0$ such that $x + \varepsilon \mathbf{B}(X) \subset C$. For any k such that $\|k\| < \varepsilon$, we have $\frac{k}{2} = \frac{-x}{2} + \frac{x+k}{2} \in C$, since C is solid and convex. Hence $\frac{\varepsilon}{2} \mathbf{B}(X) \subseteq C$. Since C is bounded, we can then define an equivalent norm, with $\|y\|_C = \inf\{\lambda > 0 : y \in \lambda C\}$. Since C is solid, $\|y\|_C = \| \, |y| \, \|_C$, and the norm is consistent with the order. Finally, $\| \cdot \|_C$ is equivalent to $\| \cdot \|$, since for all $y \in X$, we have that $\frac{\varepsilon}{2} \|y\|_C \leq \|y\| \leq \|y\|_C$. The conclusion follows by Theorem 2.5.1. $\square$

While infinite dimensional reflexive lattices contain uncountably many order extreme points, [64] gives examples of infinite dimensional lattices with finitely many order extreme points in their unit balls. In particular, X has finitely many order extreme points which are mutually disjoint if and only if X is lattice isometric to $C(K_1) \oplus_1 C(K_2) \oplus_1 ... \oplus_1 C(K_n)$ for suitable non-trivial compact Hausdorff spaces $K_1, ..., K_n$. The next results involve characterizations of finite dimensional lattices in terms of order extreme points.

Lemma 2.5.3. *Let X be a finite dimensional lattice. Then the following are equivalent:*

1. *$OEP(\mathbf{B}(X))$ is finite.*

2. *$EP(\mathbf{B}(X))$ is finite.*

3. *$EP(\mathbf{B}(X^*))$ is finite.*

Proof. (1) is equivalent to (2) by finite dimensionality and Theorem 19.2 in [64]. To show that (2) implies (3), suppose $\mathbf{B}(X)$ has finitely many extreme points. Then by Theorem 16 in [29], it is the intersection of finitely many closed half-spaces. Let $f_1, ..., f_m \in \mathbf{S}(X^*)$ such that $\mathbf{B}(X) = \{x \in X : f_i(x) \leq 1 \text{ for all } 1 \leq i \leq m\}$. Then $\|x\| = \max f_i(x)$ for all $x \in X$, so $B(X^*) = CH\{f_1, ..., f_m\}$. Otherwise, if $g \in \mathbf{S}(X^*) \setminus CH\{f_1, ..., f_m\}$, by the Hahn-Banach separation theorem there exists some $x \in \mathbf{S}(X)$ such that

$$\sup_i f_i(x) < g(x).$$

By Milman's theorem, all the extreme points of $\mathbf{B}(X^*)$ are contained in $\{f_1, ..., f_m\}$, so $\mathbf{B}(X^*)$ has finitely many extreme points. By reflexivity of finite dimensional lattices, (3) implies (2) as well. □

Theorem 2.5.4. *Let X be a finite dimensional Banach lattice. Then for all $C > 1$, there exists a C-embedding from X into an $\ell_\infty^m(\ell_1^M)$ space for some m, with $M = \dim X$. If, furthermore, X has finitely many order extreme points, then X embeds isometrically into $\ell_\infty^m(\ell_1^M)$ space for some m.*

Proof. Suppose $\{x_1^*, ..., x_m^*\}$ is an ε-net on $\mathbf{S}(X^*)_+$, where $\frac{1}{C} < 1 - \varepsilon$. Then for all $x \in \mathbf{S}(X)$, we have $\frac{1}{C} < 1 - \varepsilon \leq \sup_i x_i^*(|x|) \leq 1$. Now X^* is also finite and is thus generated by its atoms, which are the evaluation functionals e_i^* for the atoms $e_i \in X$, with $1 \le i \le M$. That is, if $x = \sum_j c_j e_j$, then $e_i^*(x) = c_i$. These functionals form a basis in X^*, so we can assume $x_i^* = \sum_j a(i,j) e_j^*$, with $a(i,j) \ge 0$. Based on this, consider the lattice $\ell_\infty^m(\ell_1^M)$, and let $u(i,j) \in \ell_\infty^m(\ell_1^M)$ correspond to the j'th atom in the i'th copy of ℓ_1^M. Then let $\phi(e_j) = \sum_i a(i,j) u(i,j)$.

ϕ is a lattice homomorphism, since it is a positive linear map that maps atoms to disjoint elements. It also is a C-embedding. Indeed, let $x = \sum c_j e_j \in \mathbf{S}(X)_+$. Then

$$\phi(x) = \sum_j c_j \phi(e_j) = \sum_j \sum_i c_j a(i,j) u(i,j),$$

so

$$\|\phi(x)\| = \sup_i \sum_j |c_j| a(i,j) = \sup_i \sum_j a(i,j) e_j^*(x) = \sup_i x_i^*(x).$$

Thus $\frac{1}{C}\|x\| \le \|\phi(x)\| \le \|x\|$.

If $\mathbf{B}(X)$ has finitely many order extreme points, then by Lemma 2.5.3, so does the dual unit ball $\mathbf{B}(X^*)$. Let $\{x_1^*, ..., x_m^*\} = OEP(\mathbf{B}(X^*))$. Then $\|x\| = \sup_{1\le i\le m} x_i^*(|x|)$. Construct ϕ in the same way as above, and observe that $\|\phi(x)\| = \sup_i x_i^*(x) = \|x\|$. □

2.6 The solid Krein-Milman Property and the RNP

We say that a Banach lattice (or, more generally, an ordered Banach space) X has the *Solid Krein-Milman Property* (*SKMP*) if every solid closed bounded subset of X is the closed solid convex hull of its order extreme points. This is analogous to the canonical Krein-Milman Property (KMP) in Banach spaces, which is defined in the similar manner, but without any references to order. It follows from Theorem 2.2.2 that the KMP implies the SKMP.

These geometric properties turn out to be related to the Radon-Nikodým Property (RNP). It is known that the RNP implies the KMP, and, for Banach lattices, the converse is also true (see [20] for a simple proof). For more information about the RNP in Banach lattices, see [61, Section 5.4]; a good source of information about the RNP in general is [18] or [25].

One of the equivalent definitions of the RNP of a Banach space X involves integral representations of operators $T : L_1 \to X$. If X is a Banach lattice, then, by [68, Theorem IV.1.5], any such operator is regular (can be expressed as a difference of two positive ones); so positivity comes naturally into the picture.

Theorem 2.6.1. *For a Banach lattice X, the SKMP, KMP, and RNP are equivalent.*

Proof. The implications RNP $\Leftrightarrow$ KMP $\Rightarrow$ SKMP are noted above. Now suppose X fails the RNP (equivalently, the KMP). We shall establish the failure of the SKMP in two different ways, depending on whether X is a KB-space, or not.

(1) If X is not a KB-space, then [61, Theorem 2.4.12] there exist disjoint $e_1, e_2, \ldots \in \mathbf{S}(X)_+$, equivalent to the canonical basis of c_0. Then the set

$$C = \overline{\mathrm{S}\Big(\big\{\sum_i \alpha_i e_i : \max_i |\alpha_i| = 1,\, \lim_i \alpha_i = 0\big\}\Big)}$$

is solid, bounded, and closed. To give a more intuitive description of C, for $x \in X$ we let $x_i = |x| \wedge e_i$. It is easy to see that $x \in C$ if and only if $\lim_i \|x_i\| = 0$, and $|x| = \sum_i x_i$. Finally, show that $x \in C_+$ cannot be an order extreme point. Find i so that $\|x_i\| < 1/2$, and consider $x' = \sum_{j \neq i} x_j + e_i$. Then clearly $x' \in C$, and $x' - x \in X_+ \backslash \{0\}$.

(2) If X is a KB-space failing the RNP, then, by [61, Proposition 5.4.9], X contains a separable sublattice Y failing the RNP. Find a quasi-interior point $u \in Y$. By [61, Corollary 5.4.20], Y is not order dentable – that is, Y_+ contains a non-empty convex bounded subset A so that for every $n \in \mathbb{N}$, $A = \overline{\mathrm{CH}(A \backslash H_n)}$, where $H_n = \{y \in Y_+ : \|u \wedge y\| \geq \frac{1}{n}\}$.

Any KB-space is order continuous, hence by [61, Theorem 2.4.2], its order intervals are weakly compact. This permits us to use the techniques (and notation) of [17] to construct a set C witnessing the failure of the SKMP. For $f \in Y^*$, let $M(A, f) = \sup_{x \in A} |f(x)|$. For $\alpha > 0$, define the *slice* $T(A, f, \alpha) = \{x \in A : f(x) > M(A, f) - \alpha\}$. By [17] (proof of the main result – p. 96), we can construct increasing measure spaces Σ_n on $[0, 1]$ with $|\Sigma_n|$ finite, as well as Σ_n-measurable functions $Y_n : [0, 1] \to A$, $f_n : [0, 1] \to Y^*$, and $\alpha_n : [0, 1] \to (0, \infty)$ such that:

1. For any n and t, $Y_n(t) \in \overline{T(A, f_n(t), \alpha_n(t))}$.

2. (Y_n) is a martingale – that is, $Y_n(t) = \mathbb{E}^{\Sigma_n}(Y_{n+1}(t))$, for any t and n ($\mathbb{E}$ stands for the conditional expectation).

3. For any n and t, $H_n \cap T(A, f_n(t), \alpha_n(t)) = \emptyset$.

4. For any n and t, $T(A, f_{n+1}(t), \alpha_{n+1}(t)) \subseteq T(A, f_n(t), \alpha_n(t))$.

Now let $C' = \overline{\mathrm{CH}(\{Y_n(t), n \in \mathbb{N}, t \in [0,1]\})}$, then the set $C = \overline{\mathrm{S}(C')}$ (the solid hull is in X) is closed, bounded, convex, and solid. We will show that C has no order extreme points. By Theorem 2.2.2, it suffices to show that no $x \in C_+ \backslash \{0\}$ can be an extreme point of C, or equivalently, of $C_+ = C \cap X_+$.

From now on, fix $x \in C_+ \backslash \{0\}$. Note that $x \wedge u \neq 0$. Indeed, suppose, for the sake of contradiction, that $x \wedge u = 0$. Find $y' \in C' \subset Y_+$, so that $x \leq y'$. For any n, we have $y' \wedge (nu) = (y' - x) \wedge (nu) \leq y' - x$. Thus, $\|y' - y' \wedge (nu)\| \geq \|x\|$. However, u is a quasi-interior point of Y, hence $y' = \lim_n y' \wedge (nu)$. This is the desired contradiction.

Find $n \in \mathbb{N}$ so that $\|x \wedge u\| > \frac{1}{n}$. Let $I_1, ..., I_m$ be the atoms of Σ_n. For $i \leq m$, define $C_i' = \overline{\mathrm{CH}(\{Y_m(t) : m \geq n, t \in I_i\})}$, and let $C_i = \overline{\mathrm{S}(C_i')_+}$.

The sequence (Y_k) is a martingale, hence $C' = \overline{\mathrm{CH}(\cup_{i=1}^m C_i')}$. Thus, by Proposition 2.2.3,

$$C = \overline{\mathrm{S}(C')} = \overline{\mathrm{S}(\overline{\mathrm{CH}(\cup_{i=1}^m C_i')})} = \overline{\mathrm{S}(\mathrm{CH}(\cup_{i=1}^m C_i))}.$$

By [17, Lemme 3], $\mathrm{CH}(\cup_{i=1}^m C_i)$ is closed. This set is clearly positive-solid, so by norm continuity of $|\cdot|$, $\mathrm{S}(\mathrm{CH}(\cup_1^m C_i))$ is closed, hence equal to C. In particular, $C_+ = \mathrm{CH}(\cup_{i=1}^m C_i)$. Therefore, if x is an extreme point of C_+, then it must belong to C_i, for some i. We show this cannot happen.

If $y \in \mathrm{S}(C_i')_+$, then we can find $y' \in C_i'$ with $y \leq y'$. By parts (1) and (4), $C_i' \subseteq \overline{T(A, f_n(t), \alpha_n(t))}$ for $t \in I_i$. By (3), $\|z \wedge u\| < \frac{1}{n}$ for any $z \in T(A, f_n(t), \alpha_n(t))$, hence, by the norm continuity of lattice operations, $\|y' \wedge u\| \leq \frac{1}{n}$. This implies $\|y \wedge u\| \leq \frac{1}{n}$. By the triangle inequality,

$$\|x \wedge u\| \leq \|y \wedge u\| + \|x - y\| \leq \frac{1}{n} + \|x - y\|.$$

hence $\|x - y\| \geq \|x \wedge u\| - \frac{1}{n}$. Recall that n is selected in such a way that $\|x \wedge u\| > \frac{1}{n}$. As $C_i = \overline{\mathrm{S}(C_i')_+}$, it cannot contain x. Thus, C witnesses the failure of the SKMP. $\square$

Chapter 3

Descriptive complexity of classes of Banach lattices

This chapter explores the descriptive complexity of various classes of Banach lattices. The additional structure of Banach lattices, combined with the enables many new ways of understanding kinds of spaces. Many of the classes involve relationships between order and norm, or between lattice operations and norm, etc. Specifically, we want to examine whether or not a class is Borel, analytic, or co-analytic (or of some higher complexity). There are some motivations tied to such categorizations, for example, related to universality. If a class is Borel, this implies that the class itself can be understood as an standard Borel space. The descriptive complexity of various equivalence relations can also serve as a rough indication of how "nice" an equivalence relation is. We begin with examples of Borel classes, and then proceed to some results about non-Borel classes.

3.1 Convexity and concavity

Let X be a separable Banach lattice, $x_1, ..., x_n \in X$, $p \in [1, \infty]$. It turns out that using Yudin-Krivine functional calculus, we can meaningfully define the expression $(\sum_1^n |x_i|^p)^{1/p}$ in the Banach lattice setting (see [43] or [55]). One such definition in particular is useful for us:

$$(\sum_1^n |x_i|^p)^{1/p} = \bigvee\{\sum_1^n a_i|x_i| : \sum_1^n a_i^q = 1\},$$

where $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p = \infty$, then we use instead $\vee_1^n |x_i|$. Note that the map $g(x)$ sending $(x_1, ..., x_n)$ to $(\sum |x_i|^p)^{1/p}$ is continuous for all n. We thus say that X is **p-convex** if there exists $M \in \mathbb{N}$ such that for all $x_1, ..., x_n \in X, n \in \mathbb{N}$

$$\Big\|\big(\sum_1^n |x_n|^p\big)^{1/p}\Big\| \leq M\big(\sum_1^n \|x_n\|^p\big)^{1/p}$$

Similarly, X is **q-concave** if there exists $M \in \mathbb{N}$ such that

$$\big(\sum_1^n \|x_n\|^q\big)^{1/q} \leq M\Big\|\big(\sum_1^n |x_n|^q\big)^{1/q}\Big\|$$

Given $p \in [1, \infty)$, let $g(p, x) = (\sum_1^n |x_i|^p)^{1/p}$ and $g(\infty, x) = \vee_1^n |x_i|$ if $p = \infty$.

Theorem 3.1.1. *For $p \in [1, \infty]$, the classes CC_p and CV_p of p-convex and p-concave lattices are Borel.*

Proof. Observe first that since $g(p, x)$ is continuous, quantification over closed images of g remain implies that the set of p-convex and q-concave lattices are also Borel. For $M \in \mathbb{N}$, we let

$$F \in CV_p(M) \iff F \in BL \bigwedge \forall m, n \in \mathbb{N}\ \|g(p, \psi_m^n(F))\| \leq Mg(p, \|\psi_m^n(F)\|),$$

Similarly, we have

$$F \in CC_p(M) \iff F \in BL \bigwedge \forall n, m \in \mathbb{N}\ g(p, \|\psi_m^n(F)\|) \leq M\|g(p, \psi_m^n(F))\|,$$

where $\psi_m^n : \mathcal{F}(\mathcal{U}^n) \to \mathcal{U}^n$ are the Borel functions induced by the Kuratowski-Ryll-Nardsewski Theorem as described in Section 1.3. The inequalities in the relations are closed and quantified over a countable set, so the Borelness of $CV_p(M)$ follows. Here, M serves the role of the constant for X that indicates the level of convexity. Thus X is p-convex iff $X \in \cup_M CV_p(M) = CV_p$. A similar argument can be made for CC_q. $\square$

We can use convexity and concavity to designate isomorphically or isometrically L_p spaces. Recall that for all $p \in [0, \infty)$, we say that X is **isomorphically $\mathbf{L_p}$** if there exists $M \in \mathbb{N}$ such that for all $x_1, ..., x_n \in X_+$ disjoint, we have

$$\frac{1}{M}\Big\|\sum_{i=1}^n x_n\Big\|^p \leq \sum_{i=1}^n \|x_i\|^p \leq M\Big\|\sum_{i=1}^n x_i\Big\|^p.$$

For $M = 1$, we simply say that X is L_p. Similarly, X is an **AM-space** if there is some M such that for all mutually disjoint $x_1, ..., x_n \in X_+$,

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq M \max(\|x_1\|, ..., \|x_n\|).$$

By [55, Lemma 1.b.13], we have that isomorphically L_p spaces are actually lattice isomorphic to L_p spaces. Observe also that for disjoint $x_1, ..., x_n$, we actually have $(\sum |x_i|^p)^{1/p} = \sum_{i=1}^{n} |x_i|$ for any $p \in [1, \infty]$. It thus follows that if X is both p-convex and p-concave, X is isomorphically L_p, and isometric to an L_p space if the p-concavity and p-convexity constant are both 1 (see also some discussion on this in [55, Section 1.d]). We thus have:

Corollary 3.1.2. *The classes $L_p^{\sim}$ and L_p of isomorphically and isometrically L_p lattices, respectively, are Borel. Similarly, $AM^{\sim}$ and AM are also Borel.*

Proof. In the isometric case, using the notation from the proof of Theorem 3.1.1, this follows from the fact that $L_p = CC_p(1) \cap CV_p(1)$. For the isomorphic case, we have $L_p^{\sim} = CC_p \cap CV_p$. Then same can be done for AM and $AM^{\sim}$. □

3.2 Order continuity

We now proceed to show that order continuity is a Borel condition. In order to do this, we first introduce the following:

For $X \in BL, x \in X, \lambda \in \mathbb{Q}, n \in \mathbb{N}$ define Λ by

$$\begin{aligned} \Lambda(X, x, \lambda, n) \iff & x \in \mathbf{B}(X)_+ \bigwedge \exists x_1, ..., x_n \in \mathbf{B}(X)_+ \\ & \left(\wedge_{i \neq j} x_i \perp x_j \bigwedge x_i \leq x \bigwedge_i \|x_i\| > \lambda \right) \end{aligned}$$

That is, $\Lambda(X, x, \lambda, n)$ holds iff $x \in X_+$ with $\|x\| \leq 1$, and there exist mutually disjoint $x_1, ..., x_n \in B(X)_+$ such that for each i, $x_i \leq x$ and $\|x_i\| \geq \lambda$.

The following theorem by Oikhberg provides a local characterization of order continuity. The proof is included here for the sake of completeness:

Theorem 3.2.1. *(Oikhberg) A lattice X is not order continuous iff there exists a λ and an x such that for all n, $\Lambda(X, x, \lambda, n)$.*

Proof. Suppose first that X is not order continuous. Recall (see e.g. [61, Section 2.4]) that a Banach lattice X is order continuous iff every disjoint order bounded sequence is norm null, so find $m \in \mathbb{N}$, $x \in X_+$ and (x_i) disjoint with $0 \leq x_i \leq x$ such that $\|x_i\| \geq 1/m$. This immediately implies that $\Lambda(X, x, 1/m, n)$ for all n.

To establish the reverse implication, suppose, for the sake of contradiction, that X is order continuous, yet there exists $x \in X_+$ and $\varepsilon > 0$ so that, for any $n \in \mathbb{N}$, there exist mutually disjoint $x_{1n}, \ldots, x_{nn} \in [0, x]$ so that $\min_i \|x_{in}\| > 2\varepsilon$. We shall achieve the desired contradiction by constructing mutually disjoint $y_1, y_2, \ldots \in [0, x]$ so that $\|y_i\| \geq \varepsilon$ for any i. Our construction will be recursive: for any n we find mutually disjoint $y_{1n}, \ldots, y_{nn}$ so that: (i) $\min_i \|y_{in}\| \geq (1 + 2^{-n})\varepsilon$, and (ii) for $i < n$, $\|y_{in} - y_{i,n-1}\| \leq \varepsilon/2^n$ (in fact, the construction also gives $y_{in} \leq y_{i,n-1}$). Then we take $y_i = \lim_n y_{in}$.

By considering the sublattice of X generated by x and x_{in} ($1 \leq i \leq n < \infty$), we can and do assume that X is separable. Then by [55, Theorem 1.b.14] we can identify X with a Banach (also called Köthe) function space on some probability measure space (Ω, μ). For future use, note the "uniform integrability": for any $\sigma > 0$ there exists $\delta > 0$ so that $\|x\mathbf{1}_A\| < \sigma$ whenever $A \subset \Omega$ satisfies $\mu(A) < \delta$.

Select y_{11} in an arbitrary manner – for instance, we can take $y_{11} = x_{11}$. Now suppose $y_{1n}, \ldots, y_{nn}$ have already been selected; pick $y_{1,n+1}, \ldots, y_{n+1,n+1}$ to satisfy our conditions. First find $m \in \mathbb{N}$ so that $\|x\mathbf{1}_A\| < \varepsilon/2^{n+1}$ whenever $\mu(A) \leq 1/m$. As $x_{1m}, \ldots, x_{mm}$ are mutually disjoint, we find $j \in \{1, \ldots, m\}$ so that $\mu(\mathrm{supp}(x_{jm})) \leq 1/m$. For $1 \leq i \leq n$, take $y_{i,n+1} = y_{in} \cdot \mathbf{1}_{\Omega \setminus \mathrm{supp}(x_{jm})}$; set $y_{n+1,n+1} = x_{jm}$. One can check that this selection works.

□

From there, the Borelness of order continuity follows:

Corollary 3.2.2. *Given a separable lattice X, the set of order continuous sublattices of X is Borel.*

Before we give the proof, we first introduce a collection of functions $\delta^m : X_+^m \to X_+^m$. For each $k \leq m$, we let $\delta_k^m(\vec{x}) = x_k - x_k \wedge (\vee_{i \neq k} x_i)$, and let $\delta^m = (\delta_k^m)_k$. These functions are all continuous, and they map positive elements to positive mutually disjoint elements. When quantifying over the countable dense sets generated by the maps ψ_k^m, we cannot guarantee these elements are disjoint, but we can work with disjoint elements by selecting almost disjoint elements from the countable set and "disjointifying" them. This approach will appear later in various chapters.

Proof of Corollary 3.2.2. We prove this by constructing an equivalent relation that is quantified only on countably sets. Let $E \in BL(X)$, and suppose x, λ is a witness to non-order continuity of E. Fix $0 < \varepsilon < \lambda/2$ and find m such that $y_m := \psi_m(\mathbf{B}(E)_+)$ with $\|y_m - x\| < \varepsilon$. It follows that for all $n, \Lambda(E, y_m, \frac{\lambda}{2}, n)$ holds. Indeed, fix m, and choose $x_1, ..., x_m$ satisfying the condition Λ for x. Since $x_i \leq x$ we have that $\|x_i - x_i \wedge y_m\| < \varepsilon$ and $\|x_i \wedge y_m\| > \frac{\lambda}{2}$. Hence for all n we have $\Lambda(E, y_m, \frac{\lambda}{2}, n)$.

Now assume $x_1, ..., x_n \leq y_m$ satisfy the Λ condition for $\frac{\lambda}{2}$. We will also show that in this instance, we need only to consider countably many potential witnessing elements. Pick $z = \psi_k^n(\bar{\mathrm{S}}_E(y_m)_+)$ and $\varepsilon < \min_i(\|x_i\| - \frac{\lambda}{2})$, such that $\|x_i - z_i\| < \frac{\varepsilon}{2}$ and for all $1 \leq i \leq n$ and $\|z_i - \delta_i^n(z)\| < \frac{\varepsilon}{2}$. Now $\delta_i^n(z) \perp \delta_j^n(z)$ for all $i \neq j$, and for each i, $\|\delta_i^n(z)\| > \lambda$. So we can assume without loss of generality that $x_1, ..., x_n$ can be constructed from the points of the form $\psi_k^n(\bar{\mathrm{S}}_E(\{y_m\})_+)$.

Hence we have that

$$\Lambda(E, y_m, \lambda, n) \iff \exists k \in \mathbb{N} \left(\bigwedge_i \|\delta_i^n(\psi_k^n(\bar{\mathrm{S}}_E(\{y_m\})_+))\| > \lambda \right).$$

This implies that Λ is itself Borel. Since we were able to find an element in the countable dense set satisfying Λ, we thus have that

$$E \text{ is order continuous } \iff \exists m \in \mathbb{N}\ \exists \lambda \in \mathbb{Q}^+\ \forall n\ \Lambda(E, \psi_m(\mathbf{B}(E)_+), \lambda, n)$$

The relation above is Borel, and so we are done. □

3.3 Atomic lattices

Recall that a Banach lattice E is atomic if it is itself the band generated by its atoms. Though the definition of an atom in a lattice is easily Borel as defined in 1.3, atomicity itself as a property of a lattice remains elusive. However, under certain conditions, we can define atomicity in a Borel manner.

Recall that $E \subseteq X$ may be atomic, but the atoms in E are not necessarily atoms in X. In addition, atomic lattices can contain atomless sublattices. For example, Let (s_n) be a dense subset of Δ, and $X = \{f + g : f \in C(\Delta) \text{ and } g \in c_0((s_n))\}$. X is atomic, but it also contains $C(\Delta)$, which is atomless. Now if we want to consider statements made specifically with atomic sublattices, it will be helpful to consider whether or not the relation

$$E \in \mathcal{A}t\mathcal{L}(X) \iff E \text{ is an atomic sublattice of } X.$$

is Borel. We first prove the following:

Proposition 3.3.1. *The relation $Id(I, E) \subseteq BL \times BL$ with I a closed lattice ideal in E is Borel.*

Proof. The Borel definition is straightforward:

$$Id(I, E) \iff \forall m, n \Big(\psi_m(I) \in E \bigwedge |\psi_m(E)| \wedge |\psi_n(I)| \in I \Big)$$

Hence Id is Borel. □

We now can prove:

Theorem 3.3.2. *For any separable lattice X, $\mathcal{A}t\mathcal{L}(X)$ is co-analytic. In particular, the class of separable atomic lattices $\mathcal{A}t\mathcal{L} := \mathcal{A}t\mathcal{L}(\mathcal{U})$ is co-analytic.*

Proof. To show $\mathcal{A}t\mathcal{L}(X)$ is co-analytic, we prove that the complement in $BL(X)$ is analytic. The way to show that $E \notin \mathcal{A}t\mathcal{L}(X)$ is to show that there exists some element in E that is atomless. For this, it is equivalent to show that there is an ideal I in E with $I \in \mathcal{A}_0(E)$, but by showing there is an ideal that is atomless, we then know that the elements in the ideal cannot be generated by atoms since ideals are closed by downward order. Hence the existence of such an ideal implies the existence of such elements. Conversely, an atomless element implies the principal ideal is atomless as well. From there, we have

$$E \notin \mathcal{A}t\mathcal{L}(X) \iff \exists I \Big(I \in BL(X) \bigwedge Id(I, E) \bigwedge I \in \mathcal{A}_0 \Big)$$

By Propositions 1.3.1 and 3.3.1 and Theorem 1.3.2, the inner relation is Borel, so $\neg\mathcal{A}t\mathcal{L}(X)$ is analytic, hence $AtL(X)$ is co-analytic. □

Remark 3.3.3. Any separable sublattice $E \subseteq X$ contains a quasi-interior point. Sometimes it is helpful to generate a quasi-interior point from a set of elements, or in the atomic case, to generate a weak unit created by summing up atoms. We first note that since X is polish, $\mathbf{S}(X)_+^{\mathbb{N}}$, equipped with the product topology, is also Polish. We then consider the map $u : \mathbf{S}(X)_+^{\mathbb{N}} \to X$, with $u((x_i)_i) = \sum 2^{-i} x_i$. This map is continuous, since in the domain, the x_i's are bounded distance from each other. We can also consider the map $\mathbf{u} : E \mapsto u(\ (\psi_i(\mathbf{S}(E)_+))_i\ )$, which takes a lattice to a "canonical" quasi-interior point.

We now consider the descriptive complexity of atomic lattices under certain conditions:

Theorem 3.3.4. *If $\mathcal{B}$ is a Borel (analytic) class of lattices with weak Fatou norms, then $\mathcal{B} \cap \mathcal{A}t\mathcal{L}$ is also Borel (analytic).*

Proof. By Theorem 3.3.2, $\mathcal{A}t\mathcal{L}$ is co-analytic, so if $\mathcal{B}$ is co-analytic, then $\mathcal{B} \cap \mathcal{A}t\mathcal{L}$ is also co-analytic. It remains to show that whenever $\mathcal{B}$ is analytic, $\mathcal{B} \cap \mathcal{A}t\mathcal{L}$ is also analytic. Thus if $\mathcal{B}$ is Borel, then $\mathcal{B} \cap \mathcal{A}t\mathcal{L}$ is both analytic and co-analytic, and thus Borel by Souslin's theorem.

To show that $\mathcal{A}t\mathcal{L} \cap \mathcal{B}$ is analytic, we prove that for all $E \in BL$,

$$E \in \mathcal{A}t\mathcal{L} \cap \mathcal{B} \iff E \in \mathcal{B} \bigwedge \exists M\ \exists (x_i)_i \in \mathbf{S}(\mathcal{U})_+^{\mathbb{N}} \Bigg[\bigwedge_{i \neq j} x_i \neq x_j \bigwedge_i \mathcal{A}(x_i, E) \tag{3.1}$$

$$\bigwedge \forall m\ \forall k\ \exists N\ \forall n \geq N \tag{3.2}$$

$$\left(\max(0, \|\psi_m(E_+)\| - M\|\psi_m(E_+) \wedge (nu((x_i)_i))\|) < \frac{1}{k} \right) \Bigg] \tag{3.3}$$

Everything inside the brackets describes Borel relations, and it is being quantified existentially, so the relation taking up lines 3.1 through 3.3 is analytic. Now if $E \in \mathcal{B}$ is atomic and has a weak Fatou norm, we let (x_i) be the sequence of atoms. Lines 3.2 and 3.3 show that the some sequence of atoms in E generate E itself as a band. In particular, Line 3.3 creates a weak unit from the atoms of E, so for any positive x we have $nu(x_i) \wedge x \uparrow x$. Finally, we pick M such that E has an M-Fatou norm, thus it satisfying the relation on the right hand side.

Suppose $E \notin \mathcal{B} \cap \mathcal{A}t\mathcal{L}$. If $E \notin \mathcal{B}$, then we are done. Otherwise let $E \in \mathcal{B} \backslash \mathcal{A}t\mathcal{L}$. Now let $M \in \mathbb{N}$, and let $x \in \mathbf{S}_E(\mathcal{U})_+$ be disjoint from all atoms in E, and choose sufficiently small $\varepsilon < \frac{1}{2}$ and $m \in \mathbb{N}$ such that $\|\psi_m(E_+) - x\| < \frac{\varepsilon}{M}$. From this, it follows that $\|\psi_m(E_+)\| > 1 - \frac{\varepsilon}{M} \geq 1 - \varepsilon$. Since x is atomless, it is disjoint to the band generated by the atoms of E. Now $u((x_i)_i)$ generates the same band, and if for some

n, $\|\psi_m(E_+) \wedge (nu(x_i))\| \geq \frac{\varepsilon}{M}$, then $\|\psi_m(E_+) - x\| \geq \frac{\varepsilon}{M}$, a contradiction. Hence for all n,

$$M\|\psi_m(E_+) \wedge (nu(x_i))\| < \varepsilon,$$

which means that for all n,

$$\|\psi_m(E_+)\| - M\|\psi_m(E_+) \wedge (nu((x_i))\| > 1 - 2\varepsilon.$$

This means that E does not satisfy the relation.

□

The following corollaries involve applications of the theorem. The first of these is useful:

Corollary 3.3.5. *The class $\mathcal{A}t\mathcal{L}_{oc} \subseteq BL$ of order continuous atomic lattices is Borel. In particular, if X is order continuous, then $\mathcal{A}t\mathcal{L}(X)$ is Borel.*

Proof. The first statement follows from the fact that order continuous lattices have a Fatou norm, while the second in addition follows from the fact that any sublattice of an order continuous lattice is also order continuous. □

Corollary 3.3.6. *The lattice isomorphism and lattice isometry classes $< \ell_p >_\sim$ and $< \ell_p >_\cong$ are Borel, for all $1 \leq p < \infty$.*

Proof. For the isometry case, this follows from the fact that $< \ell_p >_\cong = L_p \cap \mathcal{A}t\mathcal{L}$. If E is L_p, then E is lattice isometric to an $L_p(\mu)$ space (and is thus Fatou), so using Theorem 3.3.4 and the fact that any atomic separable L_p space is isometric to ℓ_p, we are done. For the isomorphic case, we simply note that $< \ell_p >_\sim = L_p^\sim \cap \mathcal{A}t\mathcal{L}$. Then again, using corollary 3.3.4, as well as a parallel result stating that atomic separable L_p spaces are isomorphic to ℓ_p, we are done. □

Remark 3.3.7. The Banach space version of this theorem for $1 < p < \infty$ was proven by Godefroy in [41] using some type-cotype arguments with the isomorphism case. However, the question remains open for the case of $p = 1$. In the case of isometry, it is still unknown whether there exists any Borel isometry class that is Borel besides l_2.

Corollary 3.3.8. *The isomorphism and isometry classes $< c_0 >_\sim$ and $< c_0 >_\cong$ are Borel.*

Proof. For the isomorphism case, it is sufficient to state that E is an AM-space by requiring that E be ∞-convex for some constant M, and also that E be also atomic order continuous. By Theorem 3.1.1 and Corollary 3.3.5, this is a Borel relation. By [55, Proposition 1.d.8], we have that E is lattice isomorphic to c_0. For the isometry case, we only need to additionally require that the ∞-convexity constant be 1. $\square$

Remark 3.3.9. The Borelness of the isomorphism class of c_0 as a Banach lattice contrasts the complete analycity of the isomorphism class of c_0 as a mere Banach space, as proven by [50].

The above results allow us to characterize the general character of a Banach lattice with respect to its atoms. The following results allow us to refer to atoms in a Borel way under certain conditions. This will allow for even more characterizations. First, we consider, for each $E \in BL$ the set of atoms $\mathcal{A}_E$. These are slices of the Borel set $\mathcal{A} \subseteq \mathcal{U} \times BL$ as described in Section 1.3. We have the following lemma:

Lemma 3.3.10. *For each $E \in BL$, the set of atoms $\mathcal{A}_E$ is closed.*

Proof. Given any two distinct atoms $e_i \neq e_j$, $e_i \perp e_j$, hence $\|e_i - e_j\| = \||e_i - e_j|\| = \|e_i + e_j\| \geq 1$, since $\|\cdot\|$ is a lattice norm. From this it follows that $\mathcal{A}_E$ is closed, since any given Cauchy sequence $(x_i) \in \mathcal{A}_E$ is eventually constant.

$\square$

What about the map $E \mapsto \mathcal{A}_E$? It turns out, this map is also Borel when its domain is $\mathcal{A}t\mathcal{L} \cap \mathcal{B}$, where $\mathcal{B}$ is a Borel class of lattices with weak Fatou norms, but it requires a bit more argumentation. Like in the last case, we cannot simply assume that $\psi_m(E)$ will be an atom. However, ψ_m will map to elements that are arbitrarily close. The idea is to define what "closeness to an atom" is in such a way that does not refer to any specific atom, and from there, we can show Borelness in a way similar to the theorem above.

Definition 3.3.11. *An element $y \in \mathbf{S}(E)_+$ is ε-**atomic in** E if for all $x \in E_+$, there exists $r \in [0, 1]$ such that $\|x \wedge y - ry\| < \varepsilon$.*

Note that if $\|y - e\| \leq \delta$ for some atom e and $y \geq 0$, then y is 2δ-atomic. Indeed, for any $x \in E_+$, $\|x \wedge y - x \wedge e\| < \delta$. Since e is an atom, $x \wedge e = re$ for some $0 \leq r \leq 1$. Furthermore $\|ry - re\| \leq r\delta \leq \delta$, so by the triangle inequality, we have $\|x \wedge y - ry\| \leq \|x \wedge y - x \wedge e\| + \|ry - re\| < 2\delta$. Now suppose that y is ε-atomic. How close is y to an atom? It turns out that if E is atomic, given a small a small enough ε, the connection between ε-atomicity and atomicity resembles a "Lipschitz" condition. More specifically:

Lemma 3.3.12. *Suppose E has an weak Fatou norm, and let M be the Fatou constant. Then if $y \in E$ is ε-atomic in E, and $\varepsilon \leq \frac{1}{13M}$, then there exists an atom $e_i \in E$ such that $\|y - e_i\| < 4\varepsilon$.*

Proof. Suppose that y is ε-atomic in E. Since E itself is atomic, y can be represented as $\sum_i r_i e_i$, where (e_i) is the sequence of atoms in E. For notational ease, let $S_n = \sum_1^n r_i e_i$, and let $T_n = \sum_{i=n+1}^{\infty} r_i e_i = y - S_n$. Now for each i, $e_i \wedge y = r_i e_i$. Since y is ε-atomic, for each i there exists r_i' such that $\|r_i e_i - r_i' y\| \leq \varepsilon$, so by the triangle inequality $|r_i' - r_i| \leq \varepsilon$.

Now I claim that there exists $r_i \geq 2\varepsilon$. Suppose not. Then for all i, $r_i < 2\varepsilon$. Since $S_n \uparrow y$ and X has a weak Fatou norm with constant M, $\sup_n \|S_n\| \geq \frac{1}{M}$. But again, by the triangle inequality, $\|S_n\|$ cannot increase significantly for each successive n, since $2\varepsilon > r_n = \|r_n e_n\| = \|S_n - S_{n-1}\| \geq \|S_n\| - \|S_{n-1}\|$. Therefore, there exists an n such that $|\|S_n\| - \frac{1}{2M}| < \varepsilon$. Pick such an n, and choose $s_1, s_2 \in \mathbb{R}$ such that $\|s_1 y - S_n\| < \varepsilon$ and $\|s_2 y - T_n\| < \varepsilon$. Hence $|s_1 - \frac{1}{2M}| < 2\varepsilon$. From this, again by triangle inequality, we have $2\varepsilon > \|(s_1 - s_2)y - (S_n - T_n)\| \geq |\|S_n - T_n\| - (s_1 - s_2)|$, but note that $S_n \perp T_n$, so $\|S_n - T_n\| = \|S_n + T_n\| = \|y\| = 1$ by assumption, so $|1 - |s_1 - s_2|| < 2e$. By similar argument, we have $2\varepsilon > \|(s_1 + s_2)y - (S_n + T_n)\| \geq |s_1 + s_2 - 1|$, so $|1 - (s_1 + s_2)| < 2\varepsilon$. Assuming $s_1 \geq s_2$, we have

$$
\begin{aligned}
1 - 2\varepsilon <& s_1 - s_2 < 1 + 2\varepsilon \\
1 - 2\varepsilon <& s_1 + s_2 < 1 + 2\varepsilon \implies \\
1 - 2\varepsilon <& s_1 < 1 + 2\varepsilon \implies \\
& |s_1 - 1| < 2\varepsilon.
\end{aligned}
$$

Yet $|s_1 - \frac{1}{2M}| < 2\varepsilon$, so $1 - \frac{1}{2M} < 4\varepsilon < \frac{4}{9M}$, which is a contradiction since $M \geq 1$. If $s_2 > s_1$, Then $|s_2 - 1| < 2\varepsilon$. Now $|(1 - s_2) - s_1| < 2\varepsilon$, so we have

$$
\begin{aligned}
-2\varepsilon <&(1-s_2)-s_1 < 2\varepsilon \implies \\
-4\varepsilon <&-s_1 < 4\varepsilon.
\end{aligned}
$$

So $s_1 < 4\varepsilon$, but since $|s_1 - \frac{1}{2M}| < 2\varepsilon$, we then have $\frac{1}{2M} < 6\varepsilon < \frac{6}{13M}$, a contradiction. Therefore, pick n such that $r_n \geq 2\varepsilon$. From this, by assumption on y we have

$$
\begin{aligned}
\varepsilon > \|r'_n y - r_n e_n\| &= \|\sum_i r'_n r_i e_i - r_n e_n\| \\
&= \||\sum_{i\neq n} r'_n r_i e_i - r_n(1-r'_n)e_n|\| \\
&= \|\sum_{i\neq n} r'_n r_i e_i + r_n(1-r'_n)e_n\| \\
&> \|r_n(1-r'n)e_n\| = r_n(1-r'_n)
\end{aligned}
$$

So we have $(1-r'_n)r_n < \varepsilon \implies (1-r'_n) < \frac{\varepsilon}{r_n} < \frac{\varepsilon}{2\varepsilon} = \frac{1}{2}$. This implies that $r'_n > \frac{1}{2}$, and $r_n > \frac{1}{2} - \varepsilon > \frac{1}{3}$ (assuming that $\varepsilon < \frac{1}{13}$). Therefore $1 \geq r'_n > 1 - 3\varepsilon > \frac{1}{2}$, and $1 \geq r_n > 1 - 4\varepsilon$. Now since $\|r'_n y - r_n e_n\| < \varepsilon$, $\|y - \frac{r_n}{r'_n}e_n\| \leq 2\varepsilon$. Therefore, we have

$$
\begin{aligned}
\|y - e_n\| \leq& \|y - \frac{r'_n}{r_n}e_n\| + \|(1 - \frac{r_n}{r'_n})e_n\| \\
\leq& 2\varepsilon + \frac{|r'_n - r_n|}{r'_n} < 2\varepsilon + \frac{\varepsilon}{r'_n} < 2\varepsilon + 2\varepsilon = 4\varepsilon.
\end{aligned}
$$

□

Now, approximate atomicity can be expressed in a Borel way by restricting our choice of constants from $[0,1]$ to $[0,1] \cap \mathbb{Q} := I_{\mathbb{Q}}$. So we define $\mathcal{E} \subseteq X \times BL \times \mathbb{R}$ in the following manner:

$$
\begin{aligned}
\mathcal{E}(x, E, \varepsilon) \iff & x \text{ is } \varepsilon\text{-atomic in } E \iff \\
& x \geq 0 \bigwedge \|x\| = 1 \bigwedge \forall m \exists q \in I_{\mathbb{Q}} \|x \wedge \psi_m(E_+) - qx\| < \varepsilon
\end{aligned}
$$

The formula above is clearly Borel, so $\mathcal{E}$ is Borel. Using this, as well as the lemma, we can now prove the following:

Theorem 3.3.13. *Let X be a Banach Lattice and $\mathcal{B}$ be a Borel class of weak Fatou sublattices of X. Then the partial map $A_{-} : \mathcal{B} \cap \mathcal{A}t\mathcal{L}(X) \to F(X)$, where $E \mapsto A_E$ is Borel.*

Proof. It is sufficient to show that the associated graph is Borel: since by Theorem 3.3.4, $\mathcal{B} \cap \mathcal{A}t\mathcal{L}$ is Borel, we claim that

$$F = \mathcal{A}_E \iff E \in \mathcal{B} \cap \mathcal{A}t\mathcal{L}(X) \bigwedge \tag{3.4}$$

$$\forall m\ \mathcal{A}(\psi_m(F), E) \bigwedge \exists M\ \forall m \in \mathbb{N}\ \forall k \in \mathbb{N}^{>13M} \tag{3.5}$$

$$\left(\mathcal{E}(\psi_m(\mathbf{S}(E)_+), E, 1/k) \implies \exists n \|\psi_m(\mathbf{S}(E)_+) - \psi_n(F)\| < \frac{4}{k}\right) \tag{3.6}$$

should work. Note that the definition resembles the statement of Lemma 3.3.12. Suppose that $F = \mathcal{A}_E$. Since $E \to \mathcal{A}_E$ is defined only in $\mathcal{B} \cap \mathcal{A}t\mathcal{L}$, we can assume E is atomic and has a weak Fatou norm, and pick M such that E satisfies the Fatou property for M. First, we note that line 3.5 implies that $F \subseteq A_E$. Recall that $\psi_m(F)$ is dense in F. Yet, from the above discussion on Lemma 3.3.10, $\|\psi_m(F) - \psi_{m'}(F)\| \geq 1$ when $\psi_m(F) \neq \psi_{m'}(F)$. Again, any Cauchy sequence in F consisting of ψ_m's are thus eventually constant, so every element in F is atomic in E, hence $F \subseteq A_E$.

To show that $\mathcal{A}_E \subseteq F$, we consider the properties expressed in line 3.6. Let e be an atom in E. Consider a Cauchy sequence $(\psi_{m_i}(\mathbf{S}(E)_+))_i := (n_i)_i$ with $n_i \to e$ such that $\|e - n_i\| < \frac{1}{i}$. From the discussion above on elements near atoms, we have that for each i, $\mathcal{E}(n_i, E, \frac{2}{i})$. By definition of F, for $i > 13M$ there exists an atom e_{k_i} of E in F such that $\|n_i - e_{k_i}\| < \frac{8}{i}$. Hence $\|e - e_{k_i}\| < \frac{10}{i}$. Now, since e, e_{k_i} are both atoms and for large enough i, we have that $\|e - e_{k_i}\| < 1$, which implies that eventually e_{k_i} is constant and $e = e_{k_i} \in F$. Hence $A_E = F$.

Finally, it is clear that if E and $\mathcal{A}_E$ satisfy the right hand side of 3.4, since every element in A_E is atomic (thus satisfying (20)), and (21) describes Lemma 3.3.12 for an appropriate M satisfying the Fatou property. So the graph of $\mathcal{A}$ is Borel, hence $\mathcal{A}$ is Borel. □

We end this section with a strengthening of Theorem 3.3.13. Given a Borel collection $\mathcal{B}$ of atomic lattices with weak Fatou norms, we want not only to find the atoms of any given $E \in \mathcal{B}$ in a Borel way, but also to be able to enumerate, again in a Borel way, the atoms in E. It is not sufficient, however to use $(\psi_m \circ \mathcal{A}_E)_m$, since it might not map to unique elements. So we construct a map $\mathbf{e} : \mathbb{N} \times \mathcal{B} \cap \mathcal{A}t\mathcal{L}(X) \to X$, where $(k, E) \mapsto \mathbf{e}_{k,E} \in A_E$, such that $\mathbf{e}(\cdot, E)$ is a bijection between $\mathbb{N}$ and A_E for all $E \in \mathcal{A}t\mathcal{L}(X)$.

Lemma 3.3.14. *Suppose $\mathcal{B}$ is a Borel class of weak Fatou lattices, and let X be a lattice. Then there is a function $\mathbf{e} : \mathbb{N} \times \mathcal{B} \cap \mathcal{A}t\mathcal{L}(X) \to X$ as described above which is Borel.*

Proof. We show that the graph is Borel:

$$\mathbf{e}_{k,E} = u \iff \big(E \in \mathcal{B} \cap \mathcal{A}t\mathcal{L}(X) \bigwedge u \in E\big) \bigwedge \Big[k = 1 \implies u = \psi_1(A_E) \bigwedge \tag{3.7}$$

$$k > 1 \implies \exists n \Big(u = \psi_n(A_E) \bigwedge \forall m < n \;\; \psi_m(A_E) \neq u \bigwedge \tag{3.8}$$

$$\exists s \in \overline{n-1}^{k-1} \big(\wedge_{i \neq j} \psi_{s_i}(A_E) \neq \psi_{s_j}(A_E) \wedge_{l<k} \vee_i \psi_l(A_E) = \psi_{s_i}(A_E)\big)\Big)\Big], \tag{3.9}$$

where $\overline{n-1} := \{1, 2, ...n-1\}$. The function $\mathbf{e}$ maps $(1, E)$ to $\psi_1(A_E)$, and (k, E) to $\psi_n(A_E)$ where n is the least number such that there are $k-1$ unique atoms among the set $\{\psi_1(A_E), ..., \psi_{n-1}(A_E)\}$. This ensures the map surjects onto A_E. All quantifiers are bounded or countable, and the relations are Borel under the above assumptions. Hence $\mathbf{e}$ is Borel.

□

3.4 Rearrangement invariant lattices

In this section, we focus on the class of rearrangement invariant order continuous atomic lattices. It turns out they are descriptive set theoretically "nice" too. In particular, we define isomorphically and isometrically invariant lattices, and show that these classes can be characterized in a Borel way, as well as their isomorphism and isometry equivalence relations. We then show that the same can be done for finite sums of rearrangement invariant lattices, which far expands on what sorts of lattices, up to isometry or isomorphism, can be easily characterized.

3.4.1 Isometrically rearrangement invariant lattices

Definition 3.4.1. *An atomic order continuous Banach lattice E is* ***rearrangement invariant*** *(or* ***r.i.*** *for short) if all permutations of its atoms are isometries (or isomorphisms.)*

Examples of r.i. lattice include ℓ_p and c_0, but they also include Orlicz sequence spaces.

Theorem 3.4.2. *The class of isometrically r.i. lattices is Borel.*

Proof. Consider the set $Q_n \subseteq \mathcal{A}t\mathcal{L}_{oc}$, where

$$Q_n(E) \iff E \in \mathcal{A}t\mathcal{L}_{oc} \bigwedge \forall n \in \mathbb{N}\ \forall \sigma \in S_n,\ q \in \mathbb{Q}^n \\ \left(\| \sum_m q_m \mathbf{e}_{m,E} \| = \| \sum_m q_m \mathbf{e}_{\sigma(m),E} \| \right)$$

By Theorem 3.3.13, the inner relation is Borel. Q_n states namely that all permutations of the first n atoms are isometric on n-supported vectors.

Now define $R \subseteq \mathcal{A}t\mathcal{L}$ by

$$R(E) \iff \bigcap_n Q_n(E).$$

Since E is order continuous, it is enough to check through the finite permutations, since each element is approximated by the linear span of E's atoms. Therefore, E is r.i. $\iff R(E)$ holds, and R is Borel, the class of isometrically r.i. lattices is Borel. □

Theorem 3.4.3. *Let E be an r.i. order continuous sublattice. Then the isometry class of E is Borel. More generally, The isometry equivalence relation $I_{\cong} \subseteq R_{\cong} \times R_{\cong}$, where $R_{\cong}$ is the Borel set of all r.i. order continuous sublattices, with*

$$I_{\cong}(E, E') \iff E \text{ and } E' \text{ are lattice isometric}$$

is a Borel equivalence relation.

Proof. Again, we just use a Borel formula and claim that:

$$I_{\cong}(E,E') \iff E',E \in \mathcal{A}t\mathcal{L}_{oc} \bigwedge \forall n \in \mathbb{N}, q \in \mathbb{Q}^n \tag{3.10}$$

$$\left(\| \sum_1^n q_m \mathbf{e}_{m,E} \| = \| \sum_1^n q_m \mathbf{e}_{m,E'} \| \right) \tag{3.11}$$

Again, from the above, if E and E' are isometric, then clearly the right hand side holds. Conversely, if, all finitely supported vectors with rational coefficients have equal norm, and since these form dense subsets of E and E', the map $\mathbf{e}_{m,E} \mapsto \mathbf{e}_{m,E'}$ extends to an isometric isomorphism.

$\square$

We now consider finite sums of r.i. lattices (ex: $\ell_p \oplus \ell_q$). In addition, we add the following natural requirement: if $x_i \in E_i$, where E_i is isometrically r.i., we add that the norm of $\sum x_i$ is invariant under permutations over the atoms within individual E_i's. That is, for all i and all isometries σ_i over E_i, we have $\| \sum x_i \| = \| \sum \sigma_i(x_i) \|$.

When a lattice $E = E_1 \oplus E_2 \oplus ... \oplus E_n$ with each E_i r.i. and isometrically invariant over permutations over each of the E_i's as described above, and n is the smallest such number where this is the case, we will state that $E \in R_n$. The key is for each step n, to generate a partition of n of finite dimensional r.i. lattices for which the finite permutations do not change the norms of **any** element supported by the first n atoms.

Theorem 3.4.4. *The set R_n of exact n-sums of r.i. sublattices of X is Borel.*

Proof. The proof will be by induction on n. For $n = 1$, this is proven by Theorem 3.4.3. Now assume that for all $m < n$, R_m is Borel. The idea is that for each $k \in \mathbb{N}$, we want to partition $\overline{k} := \{1, ..., k\}$ into n sets $C_1^k, ..., C_n^k$ such that for elements supported by atoms $\mathbf{e}_{1,E}, ..., \mathbf{e}_{k,E}$ the norm is preserved when you permute elements within each set in the partition. For indexing purposes later on, we also add the following conditions: 1) if the partition includes empty sets, then all the non-empty sets have the lowest indices, and 2) if $i < j$, then for non-empty C_i^k, C_j^k, we have $\min C_i^k < \min C_j^k$. These conditions can then be defined as a relation on $\bigcup_k \mathbb{P}(\bar{k}) \times \mathbb{N}$, where $\mathbb{P}(\overline{k})$ is the power set of $\overline{k}$, as follows:

$$sp(C_1,...,C_n,k) \iff C_1,...,C_n \subseteq \overline{k} \bigwedge 1 \in C_1 \tag{3.12}$$

$$\bigwedge_{i>1}^{n} C_i \neq \emptyset \implies (C_{i-1} \neq \emptyset \bigwedge \min(C_{i-1}) < \min(C_i)) \tag{3.13}$$

$$\bigwedge_{i\neq j} C_i \cap C_j = \emptyset \bigwedge \cup_i C_i = \overline{k} \tag{3.14}$$

For each $k \in \mathbb{N}$ we will define the set $P_{n,k} \subseteq (\bigcup_k \mathbb{P}(\bar{k}))^n \times \mathbb{N} \times AtL_{oc}(X)$, where

$$p_n(C_1,...,C_n,k,E) \iff sp(C_1,...,C_n,k) \bigwedge \forall \tau = (\sigma_i)_{i=1}^n \in S_{c_1}...S_{c_n} \tag{3.15}$$

$$\forall M \geq k \forall (q_j) \in \mathbb{Q}^M \Big(\| \sum_1^M q_j \mathbf{e}_{j,E} \| = \| \sum_1^M q_j \mathbf{e}_{\tau(j),E} \| \Big), \tag{3.16}$$

where each S_{C_i} stands for the permutation group of C_i when $C_i \neq \emptyset$, and is the trivial group otherwise, and $S_{c_1}...S_{C_n} \subseteq S_\infty$ is the permutation subgroup over $\mathbb{N}$ generated by all of the S_{C_i}'s. The last line states that for any finitely supported vector with rational coefficients, the norms do not change when a composition of permutations of each C_i is applied to the first k atoms. From here on, we say that $C_1,...,C_n$ is a **suitable n-partition of k over E** if $p_n(C_1,...,C_n,k,E)$ holds. Finally, we define the relation $P_{n,k}$ by

$$P_{n,k}(E) \iff \exists C_1,...,C_n \big(p_n(C_1,...,C_n,k,E)\big)$$

$P_{n,k}$ is Borel since it is only quantified over countable sets, and in the last line the equality of norms over atoms is Borel. Based on this, I claim that

$$E \in R_n \iff E \in \mathcal{A}t\mathcal{L}_{oc} \bigwedge_{1\leq m<n} E \notin R_m \bigwedge \forall k \in \mathbb{N},\ P_{n,k}(E)$$

If E is an exact sum of n r.i. lattices, then the above easily follows.

Now suppose that $E \in R_n$. Note that since $E \notin R_m$ when $m < n$, it follows that for some k, there is a suitable partition $C_1^k,...,C_n^k$ of k over E of with each C_i^k is by necessity non-empty.

Indeed, otherwise, we can make a tree over $\{1,...,n-1\}$ encoding suitable partitions over E whose branches are tuples $(s_1, ..., s_k)$, where for any $1 \leq l \leq k$, $s_l = i$ when $l \in C_i$. This tree has infinitely many nodes but is finitely branching, so by König's Lemma, there is an infinite branch inducing an $n-1$ partition of $\mathbb{N}$ and associated lattices $E_1, ..., E_m$ for some $m < n$. Then $E = E_1 \oplus ... \oplus E_m$ is in R_m for some $m < n$, which contradicts the initial assumption.

We can use a similar argument to conclude the proof, but we will make an even stronger claim with Lemma 3.4.5, which will be proven later: *For all $k' \geq k$, the partition satisfying $p_n(C_1^{k'}, ..., C_n^{k'}, k', E)$ is unique.*

Based on this, we can arrive at a partition of $\mathbb{N}$ that corresponds to the atoms. If $k' > k$, then consider the partition $C_1^{k'}, ..., C_n^{k'}$ of k'. Now take, for each $C_i^{k'}$, $C_i^{k'} \cap \bar{l}$, where $k < l \leq k'$. The result is a suitable n-partition of l, but by the lemma, $C_i^{k'} \cap \bar{l} = C_i^l$. Now let $C_i' = \bigcup_{l>k} C_i^l$. Then $C_1', ..., C_n'$ gives us the necessary n-partition of $\mathbb{N}$.

$\square$

Let's now prove Lemma 3.4.5, stated more precisely as follows:

Lemma 3.4.5.

Suppose $E \in R_n$. Let k be the least number such that for any suitable partition $C_1, ..., C_n$ of k over E, each C_i must be non-empty. Then for all $k' > k$, the suitable n-partition of k' is unique.

Proof. Suppose not. Let $k' \geq k$, and let $C_1, ..., C_n$ and $D_1, .., D_n$ be two different partitions satisfying the condition $P_{n,k'}$. By the above assumption, each C_i, D_i must be non-empty. If, say, $C_i = \emptyset$ for some i, restricting each C-set to its elements $\leq k$ gives a partition of k fulfilling $P_{n,k}$ with less than n elements, contradicting the assumption of minimality of k. Let j be the first index where $C_j \neq D_j$. without loss of generality, assume that there is $m \in C_j / D_j$. Note that $C_j \cap D_j \neq \emptyset$, since they have the same minimum element m' and both C_j's and D_j's are suitable partitions, so $\min C_{i'} < \min C_i$ and $\min D_{i'} < \min D_i$ whenever $i' < i$. It is also clear that $j < n$. Now since $m \notin D_j$, since the D-sets form a partition of k', $m \in D_{j'}$, where $n \geq j' > j$.

If such is the case, it follows that k' can be re-partitioned into $n-1$ non-empty sets and one empty set satisfying $P_{n,k'}$ These sets would be $D_1, ..., D_{j-1}, D_j \cup D_{j'}$, followed by the rest of the $D_i's$ not including D_j', re-indexed to satisfy the non-emptiness properties outlined in the definition of $P_{n,k}$. This is done as follows. Denote the elements of D_i by $d^i_1, d^i_2, ...d^i_{|D_i|}$, where the elements are indexed by increasing order. Then $m = d^{j'}_\alpha$ and $m' = d^j_1$.

Denote a finite permutation in $s \in S_\infty$ by $(t_0, ..., t_l)$, with $s(t_i) = t_{i+1 \mod l}$ and $s(t) = t$ if $t \neq t_i$ for all $0 \leq i \leq l$. Since the C-sets and D sets satisfy the conditions of $P_{n.k'}$, We have that the permutations consisting of $(m', d^j_2, ...d^j_{|D_j|})$, $(m, d^{j'}_{\alpha+1}, ..., d^{j'}_{|D_{j'}|}, d^{j'}_1, ..., d^{j'}_{\alpha-1})$, and (m, m') correspond to permutations of $S_{k'}$ whose corresponding operators preserve the norms of finitely supported vectors in E.

To show that a lesser repartitioning exists, it is sufficient to show that every permutation of $D_j \cup D_{j'}$ preserves norms. Consider the permutations above, and note that norms are still preserved when one takes compositions of these permutations. Hence, norms are preserved over the permutation

$$(m', d^j_2, ...d^j_{|D_j|}) \circ (m, d^{j'}_{\alpha+1}, ..., d^{j'}_{|D_{j'}|}, d^{j'}_1, ..., d^{j'}_{\alpha-1}) \circ (m, m')$$

which is $(m', d^{j'}_{\alpha+1}, ..., d^{j'}_{|D_{j'}|}, d^{j'}_1, ..., d^{j'}_{\alpha-1}, m, d^j_2, ...d^j_{|D_j|})$. This composition is a cycle σ of all the elements in $D_j \cup D_{j'}$. If $|D_j \cup D_{j'}| = 2$, we are done. If not, then either D_j or $D_{j'}$ (or both) contain more than one element. This implies that either $(m', d^j_{|D_j|})$ or $(d^{j'}_{\alpha-1}, m)$ are transpositions inducing lattice isometries on E. Either way, we also have a transposition whose elements are adjacent elements in a full cycle over $D_j \cup D_{j'}$. It is a basic result in group theory that the group of permutations of a finite set generated by a cycle over all the elements and a transposition over elements adjacent in the cycle generate the entire permutation group itself. Hence all the permutations of atoms indexed with elements $D_j \cup D_{j'}$ preserve norms. This implies a partition of k', as described above, consisting of only $n - 1$ sets, satisfying $P_{n,k}$, a contradiction. $\square$

Remark 3.4.6. The lemma implies not only the existence of a unique partition, but also that this partition can be determined in a Borel way, assuming that we know that there is a finite partition.

There is also a similar result for the isometry relation in $R_n \times R_n$, which can be shown with a sentence, albeit a more complicated one.

Theorem 3.4.7. *The isometry relation I_n on $R_n \times R_n$ is Borel. In fact, the isometry relation on finite sums of r.i. lattices is Borel.*

Proof. Note that $I(E, E')$ iff there is n such that $E, E' \in R_n$, and $I_n(E, E')$. So we just show that I_n is Borel. The aim is to find two n-partitions $D_1, ..., D_n$ and $D'_1, ..., D'_n$

of $\mathbb{N}$ and some $\sigma \in S_n$ such that the lattices generated by atoms indexed by D_i is isometric to that generated by $D'_{\sigma(i)}$. Since we are dealing with isometries, we need to ensure that $|D_i| = |D'_{\sigma(i)}|$. This can be done on the finite level by ensuring that for any given $k \in \mathbb{N}$ we have a suitable partition $C_1, ..., C_n$ of $\overline{k}$, $C_i \subseteq D_i$, some $k' \geq k$ and a suitable partition $C'_1, ..., C'_n$ of $\overline{k'}$ such that $|C_i| \leq |C'_{\sigma(i)}|$. Similarly, for any k' we want a suitable partition $C'_1, ..., C'_n$ of $\overline{k'}, C'_i \subseteq D'_i$, some $k \geq k'$ with suitable partition $C_1, ..C_n$ of $\overline{k}$, $C_i \subseteq D_i$, with $|C_i| \geq |C'_{\sigma(i)}|$. All the above, we can define in a Borel way. Let $F_\iota(X, Y)$ denote the set of all injective functions from X to Y, and consider the relation A by

$$\begin{aligned} A(E, E', k) \iff & \exists k' \in \mathbb{N},\ \sigma \in S_n,\ (C_i)_1^n P(\overline{k}),\ (C'_i)_1^n \in P(\overline{k'}) \\ & \Big[p_n((C_i)_1^n, k, E) \bigwedge p_n((C'_i)_1^n, k', E') \bigwedge \exists (f_i) \in \prod_i^n F(C_i, C'_{\sigma(i)}) \\ & \Big(\forall (q_j) \in \mathbb{Q}^k \| \sum_i^n \sum_{j \in C_i} q_j \mathbf{e}_{j,E} \| = \| \sum_i^n \sum_{j \in C_i} q_{f_i(j)} \mathbf{e}_{f_i(j),E'} \| \Big) \Big]. \end{aligned}$$

Then I claim that

$$\begin{aligned} I_n(E, E') \iff & R_n(E) \bigwedge R_n(E') \bigwedge \\ & \forall k \ \big(A(E, E', k) \bigwedge A(E', E, k) \big) \end{aligned}$$

Note that if E is isometric to E' then the result follows. Suppose that E, E' satisfy the relation. Note also then that by Lemma 3.4.5, there is some K that the n-partitions over the first K atoms satisfying p_n must be unique while satisfying p_n. Hence for all $k \geq K$, each $\overline{k}$-partition (C_i), we have $C_i \subseteq D_i$, and similarly $C'_i \subseteq D'_i$. Furthermore, the relation A describes the existence of injective functions from C_i into $D'_{\sigma(i)}$ for any k, and vice versa. The result then is that if D_i is finite, we must have $|D'_{\sigma(i)}| = |D_i|$. If D_i is infinite, then the C_i's increase in size as k increases, but you must have then that for some other $k' \geq k$ and partition (C'_i), there is an injective function from C_i to $C'_{\sigma(i)} \subseteq D'_{\sigma(i)}$ Hence $D'_{\sigma(i)}$ is also infinite. Hence we can induce a lattice isomorphism sending atoms indexed by D_i bijectively to those indexed by $D'_{\sigma(i)}$. So $E \cong E'$.

□

3.4.2 Isomorphically rearrangement invariant lattices

Definition 3.4.8. *An atomic order continuous lattice X is* ***isomorphically rearrangement invariant****, or* ***isomorphically r.i.*** *if every permutation σ over the atoms of X induces a lattice isomorphism.*

For merely isomorphically, and not isometrically r.i. spaces, some extra care is needed. If X is isomorphically r.i., then any σ is an isomorphism, but it may have a certain amount of distortion M, that is, σ is an M-embedding (see the definition of M- embeddings in Section 1.2). Given $\sigma \in S_\infty$ and $x = \sum a_i e_i \in X$, let $\sigma(x)$ be the short hand for $\sum a_i e_{\sigma(i)}$. We first show:

Theorem 3.4.9. *The class of isomorphically r.i. spaces is Borel.*

We need to first prove the following lemma:

Lemma 3.4.10. *Let X be isomorphically r.i. Then*

$$\sup_{\sigma \in S_\infty} \|\sigma\| < \infty.$$

Proof. Suppose the contrary. We will then show that X cannot be r.i, by constructing a permutation $\tau = \tau_1\tau_2...$ where the $\tau_i' s$ are mutually disjoint, finite length permutations, which is not bounded. Choose σ_1 such that $\|\sigma_1\| > 1$. By order continuity, there exists a finitely supported element $x_1 \in S(x)_+$ such that $\sigma_1(x_1) > 1$. Let τ_1 be a finite permutation of $D_1 := \text{supp}(x_1) \cup \text{supp}(\sigma_1(x_1))$ such that for all k such that e_k supports x_1, $\sigma_1(k) = \tau_1(k)$. This can be done in the following manner: for $i \in \text{supp}(x_1)$ let $\tau_1(i) = \sigma_1(i)$. Note that $|D_1 \backslash \text{supp}(x_1)| = |D_1 \backslash \text{supp}(\sigma_1(x_1))|$, so let τ_1 biject these two sets. = Then $\tau_1|_{\mathbb{N}\backslash D_1} = id_X$. Since $\tau_1(x_1) = \sigma_1(x_1)$, $\|\tau_1\| > 1$ as well. We now proceed inductively on n. Suppose $\tau_1, ..., \tau_n$ have been chosen with $\|\tau_n\| > n$, and suppose that $D_1, ..., D_n$ mutually disjoint finite sets have been chosen such that $Dom(\tau_i) = D_i$.

Now we claim that $T := \{\sigma : \sigma|_{D_i} = id \text{ for all i} \leq n\}$ remains unbounded. Let $K \in \mathbb{N}$, and suppose $\sigma' \in S_\infty$ such that $\|\sigma'\| > 3K \max(\cup_i D_i)$. Observe here that at worst, any permutation τ over the standard basis in $\mathbb{R}^k$ equipped with a Lattice norm will have distortion at most k. If $\|\sum_1^k a_i e_i\| = 1$, then $\max|a_i| \geq 1/k$ and $\sum |a_i| \leq k$, thus $1/k \leq \|\sum_1^k a_{\tau(i)} e_i\| \leq k$. Now there exists a finite permutation $\tau' \in S_{3\max(\cup D_i)}$ such that $\tau'\sigma' \in T$. As a result, $\max(\|\tau'\|, \|{\tau'}^{-1}\|) \leq 3\max(\cup_i D_i|)$, so $\|\tau'\sigma\| > K$.

Therefore, pick a $\sigma_{n+1} \in T$ such that $\|\tau_1, ..., \tau_n \sigma_{n+1}\| > n+1$. Find a finitely supported $x_{n+1} \in S(X)$ such that $\|\tau_1, ..., \tau_n \sigma_{n+1}(x_{n+1})\| > n+1$, like in the initial step, let τ_{n+1} be a finite permutation defined on $D_{i+1} := \text{supp}(x_{n+1}) \cup \text{supp}(\sigma(x_{n+1})) \setminus (\cup_i D_i)$ such that for all k such that $e_k \in D_{i+1}$ supports x_{n+1}, we have $\tau_{n+1}(k) = \sigma_{n+1}(k)$. The construction is just like in the initial case, given that $\sigma_{n+1}|_{\cup_i D_i} = id|_{\cup_i D_i}$. Since $\tau_{n+1}(x_{n+1}) = \sigma_{n+1}(x_{n+1})$, it follows that $\|\tau_{n+1}\| > n+1$. □

Proof (of theorem). We adapt the approach from Theorem 3.4.2. Let

$$Q_n(E, M) \iff E \in \mathcal{A}t\mathcal{L}_{oc} \bigwedge_{\sigma \in S_n} \bigwedge_{q_1,\dots,q_n \in \mathbb{Q}} \left(\| \sum q_i \mathbf{e}_{m,E} \| \leq M \| \sum q_i \mathbf{e}_{\sigma(m),E} \| \right)$$

Then we show that

$$R_\sim(E) \iff \bigwedge_M \bigwedge_n Q_n(E, M))$$

If we assume $R_\sim(E)$ holds, then easily E is isomorphically r.i. If E is isomorphically r.i., then by Lemma 3.4.10, the norms of the permutations are bounded by some integer M, so $R_\sim(E)$ holds. □

Note that Lemma 3.4.10 can be used to show the following (not sure if this was already known, but I include here for completion):

Corollary 3.4.11. *X is an isomorphically r.i. space iff it is lattice isomorphic to an isometrically r.i. space.*

Proof. $\Longleftarrow$ is obvious. To prove $\Longrightarrow$, we construct a norm $|||\cdot|||$ on X by

$$|||x||| = \sup_{\sigma \in S_\infty} \|\sigma(x)\|$$

By Lemma 3.4.10, there exists an M such that for all $x \in X$, $\|x\| \leq |||x||| \leq M\|x\|$, so $|||\cdot|||$ is an equivalent lattice norm. Thus $id : (X, \|\cdot\|) \to (X, |||\cdot|||)$ is a lattice isomorphism. Note also that the definition implies that for all $x \in X$ and $\sigma \in S_\infty$ we have that $|||x||| = |||\sigma(x)|||$. Thus $(X, |||\cdot|||)$ is isometrically r.i. □

Corollary 3.4.12. *If X is isomorphically r.i., then it is lattice isomorphic to each of its infinite dimensional ideals.*

Proof. Let $Y \subseteq X$ be an ideal, and let $(e_{n_k})_k$ be the atoms of X enumerating the atoms of Y. Let M be the rearrangement distortion constant for X. Then the map $\psi : e_k \mapsto e_{n_k}$ from X to Y is an isomorphism with distortion at most M. Let $x = \sum_1^m a_i e_i$ be finitely supported in X. Find a permutation $\sigma \in S_\infty$. with $\sigma(k) = n_k$

for $1 \leq k \leq m$. Then $1/M\|x\| \leq \|\sigma(x)\| \leq M\|x\|$. Thus ψ extends to a lattice isomorphism between X and Y. $\square$

If $E = \oplus_1^n E_i$ is an exact n-sum of n isometrically r.i. lattices E_i, the argument for Theorem 3.4.3 allows for the possibility of finite dimensional E_i. However, these finite dimensional sublattices cannot be isolated in isomorphically r.i. spaces. This allows for some advantages, but it also means that partitions are no longer unique.

We can say that $\oplus_1^n E_i$ is **isomorphically an n-sum of r.i. lattices** if there exists some $M \in \mathbb{N}$ such that for all $\sum_1^n x_i \in \oplus_1^n E_i$ and all permutations σ_i on the atoms in E_i, we have

$$1/M\|\sum_1^n x_i\| \leq \|\sum_1^n \sigma_i(x_i)\| \leq M\|\sum x_i\|.$$

Observe here that, as in the case of finite sums of isometrically r.i. lattices, we want permutations on individual E_i's to have at most a bounded amount of distortion on other E_i's. However, this automatically follows when one can partition E into isomorphically r.i. disjoint bands $E_1 \oplus ... \oplus E_n$. If $E = E_1 \oplus ... \oplus E_n$ with each E_n isomorphically r.i., then for any $x = \sum_{i=1}^n x_i$ with $x_i \in E_i$, we have

$$\max_i \|x_i\| \leq \|x\| \leq n \ \max \|x_i\|.$$

This means that E is lattice isomorphic to $E_1 \oplus_\infty ... \oplus_\infty E_n$, which has this property.

The proof for the Borelness of the class $R_n^\sim$ of n-sums of isomorphically r.i. lattices is similar in structure. However, the partitioning set is not unique past a certain k, since isomorphism allows for distortion, so we cannot apply Lemma 3.4.5. Thus we present a separate proof:

Theorem 3.4.13. *The class $R_n^\sim$ of lattices that are exact n-sums of isomorphically r.i. lattices is a Borel class.*

Proof. We employ a similar construction to the proof in Theorem 3.4.4. Recall the definition of $sp(C_1, ..., C_n, k, E)$ for each finite $C_i \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ in lines 3.12 - 3.14. We now define an adapted relation p_n^N, where

$$p_n^N(C_1, ..., C_n, k, E) \iff sp(C_1, ..., C_n, k) \bigwedge \tag{3.17}$$

$$\forall \tau \in S_{C_1} ... S_{C_n}, M \geq k, (q_j) \in \mathbb{Q}^M \left(\| \sum_1^M q_j \mathbf{e}_{j,E} \| \leq N \| \sum_1^M q_j \mathbf{e}_{\tau(j),E} \| \right) \tag{3.18}$$

the main switch is from equality to bounded inequality in line 3.18. We then define

$$P_{n,k}^N(E) \iff \exists C_1, ..., C_n \big(p_n^N(C_1, ..., C_n, k, E) \big)$$

As in the proof of Theorem 3.4.4, we claim that

$$R_n^{\sim}(E) \iff E \in \mathcal{A}t\mathcal{L}_{oc} \bigwedge_{1 \leq m < n} \neg R_m(E) \bigwedge \exists N \; \forall k \; P_{n,k}^N(E)$$

It should be clear that if $E \in R_n^{\sim}$, then $E = \oplus_1^n E_i$, with $E_i \in R_{\sim}$, and that n is the least such number where this is the case. Hence by induction, $E \notin R_m^{\sim}$ for any $m < n$. Let $C_1, ...C_n$ be the partition of $\mathbb{N}$ such that $C_i = \{k \in \mathbb{N} : \mathbf{e}_{k,E} \in E_i\}$. By theorem 3.4.9, we let N_i be an upper bound for distortion of E_i. Then if we let $N = \max_i N_i$, we are done.

Now suppose that E fulfills the relation on the right. Fix N satisfying the relation, and let $T_E \subseteq \overline{n-1}^{<\mathbb{N}}$ be the tree generated by all finite sequences $(a_1, ..., a_k)$ representing a partition that satisfies $p_{n,k}^N$. That is, if for $i \leq n$, we have $C_i = \{m : a_m = i\}$, then $p_n^N(C_1, ...C_n, k, E)$. Note that some of the later C_i's may be empty. By definition of *sp* in the proof of Theorem 3.4.9 (see lines 3.12 - 3.14), such sequences do indeed comprise a tree. Now note that the tree is finitely branching, and by the quantification over all $k \in \mathbb{N}$, it has infinitely many elements. By König's lemma, T_E has an infinite branch s whose elements correspond to a n-partition $\coprod C_i$ of $\mathbb{N}$. Furthermore, each set is infinitely large. This is due to the argument in the proof of Lemma 3.4.10. If, for example, C_1 were infinite and C_2 were finite, then the lattice generated by atoms indexed by elements in $C_1 \cup C_2$ would be isomorphically r.i., contradicting the claim that $E \notin R_m^{\sim}$ for some $m < n$. Finally, we let E_i be the lattice generated by atoms indexed by C_i. each E_i is isomorphically r.i with distortions bounded by N, Thus $R(E)_n^{\sim}$ follows. □

Remark 3.4.14. Note that the argument used here also works for the n-sum of isometric lattices. However, the uniqueness of isometric partition and its construction has the advantage of isolating the isometric r.i. sublattices making up the n-sum form some lattice $E \in R_n^{\cong}$. In other words, One can map some $E \in R_n^{\cong}$ to a sequence representing the partition of atoms which generate each set E_i. This uniqueness is lost in the proof above, and it is impossible in the isomorphic case.

We now proceed to give some proofs on the Borelness of isomorphism classes on isomorphically r.i. lattices, as well as finite sums of isomorphically r.i. lattices. We start with a clear fact:

Lemma 3.4.15. *Let $\phi : E \to E'$ be a lattice isomorphism between order continuous atomic lattices with distortion M, and suppose $\phi(e_i) = a_i e'_i$ for atoms $e_i \in E$ and $e'_i \in E'$. Then the map $\psi(e_i) = e'_i$ is also a lattice isomorphism with distortion M^2.*

Proof. Observe for each i, $1/M \geq a_i \geq M$, since the distortion of ϕ is at most M. Then the map $\phi' : e'_i \mapsto \frac{1}{a_i} e'_i$ induces a well-defined lattice isomorphism, and for each $x = \sum r_i e'_i$, we have $1/M \| \sum r_i e_i \| \leq \| \sum \frac{r_i}{a_i} \| \leq M \| \sum r_i e'_i \|$. Thus $\psi = \phi' \circ \phi$ has at most distortion M^2. $\square$

Theorem 3.4.16. *The isomorphism equivalence relation $I_1^{\sim}(E, E') \subseteq R_1^{\sim} \times R_1^{\sim}$ for isomorphically r.i. lattices is Borel.*

Proof. The Borel statement mirrors that of isometrically r.i. lattices:

$$I_1^{\sim}(E, E') \iff E', E \in AL_{oc} \bigcap_n \exists M \ \forall n \in \mathbb{N}, q \in \mathbb{Q}^n$$

$$\left(1/M \| \sum_1^n q_m \mathbf{e}_{m,E'} \| \leq \| \sum_1^n q_m \mathbf{e}_{m,E} \| \leq M \| \sum_1^n q_m \mathbf{e}_{m,E'} \| \right)$$

If E and E' are isomorphic, then by Lemma 3.4.15, we can assume that the isomorphism maps normalized atoms to normalized atoms, and then compose the isomorphism with the appropriate permutation of atoms in E' to map $\mathbf{e}_{m,E}$ to $\mathbf{e}_{m,E'}$. Conversely, if $I_1^{\sim}(E, E')$, all finitely supported vectors with rational coefficients are M-equivalent in norm for some M, and since these form dense subsets of E and E', the map $\mathbf{e}_{m,E} \mapsto \mathbf{e}_{m,E'}$ extends to an isomorphism of at most distortion M. $\square$

Finally, we end this section with a proof of the Borelness of the isomorphism equivalence relation for finite sums of isomorphically r.i. lattices.

Theorem 3.4.17. *The isomorphism relation on finite sums of isomorphically r.i. lattices is Borel.*

The proof combines the approaches from Theorems 3.4.13 and 3.4.16 and uses the same notation. Suppose for all $m < n$, $I_m^{\sim}$ is Borel. We first want to find two partitions $D_1, ..., D_n$ and $D_1', ..., D_n'$ such that each $E_i := E|_{D_i}$ is lattice isomorphic to $E_i' := E|_{D_i'}$. Since R_n' is defined to make n the smallest such number that allows for such a partition, we may assume that each D_i and D_i' is infinite. To this end, we fix M such that any permutation $\sigma_1 ... \sigma_n$, with each σ_i permuting D_i, induces a lattice isomorphism on E with distortion at most M, and similarly with E'. We also assume that if E, E' are isomorphic, then M also is a distortion constant for a lattice isomorphism between E and E' composed with any of their suitable permutations.

As in Theorem 3.4.13, for all $k \in \mathbb{N}$, there should exist a suitable n-partition of k with sets $C_1^k, ..., C_n^k$, such that the permutation group $S_{C_1^k} ... S_{C_n^k}$ over the atoms in E have at most distortion M. Furthermore, there should be some k' and a suitable n-partition $C_1^{k'}, ..., C_n^{k'}$ of k' so that group of permutations $S_{C_1^{k'}} ... S_{C_n^{k'}}$ over the atoms in E' have at most a distortion M. Finally, we want some $\sigma \in S_n$ such that $|C_i^k| \leq |C_{\sigma(i)}^{k'}|$ and that any map sending the atoms in E indexed by C_i^k to atoms in E' indexed $C_i^{k'}$ will induce a partial lattice isomorphism with distortion at most M. This can be expressed in a way similar to that in Theorem 3.4.13: let $A(E, E, k, M)$ be defined by:

$$A(E, E', k, M) \iff \exists k' \in \mathbb{N},\ \sigma \in S_n,\ (C_i)_1^n P(\overline{k}),\ (C_i')_1^n \in P(\overline{k'}) \tag{3.19}$$

$$\Bigg[p_n^M((C_i)_1^n, k, E) \bigwedge p_n^M((C_i')_1^n, k', E') \bigwedge \exists (f_i) \in \prod_i^n F(C_i, C_{\sigma(i)}') \Bigg(\forall (q_j) \in \mathbb{Q}^k \tag{3.20}$$

$$1/M \| \sum_i^n \sum_{j \in C_i} q_j \mathbf{e}_{j,E} \leq \| \sum_i^n \sum_{j \in C_i} q_{f_i(j)} \mathbf{e}_{f_i(j),E'} \| \leq M \| \sum_i^n \sum_{j \in C_i} q_j \mathbf{e}_{j,E} \| \Bigg) \Bigg]. \tag{3.21}$$

We also want the same relationship with E and E' switched. Then

$$I_n^{\sim}(E,E') \iff E,E' \in R_n^{\sim} \bigwedge \exists M \ \forall k$$
$$\big(A(E,E',k,M) \bigwedge A(E',E,k,M)\big).$$

If E is isomorphic to E', we can assume that the associated isomorphism ϕ sends normalized atoms to normalized atoms by Lemma 3.4.15. Let N_E and $N_{E'}$ be the rearrangement distortion constants for E and E'. Then $M = N_E\|\phi\|N_{E'}$ qualifies as a distortion bound satisfying both $A(E,E',k,M)$ and $A(E',E',k,M)$ for all k.

Suppose now that E and E' satisfy the relation. Consider the tree $T_{E,E'}$ formed by branches in $(\overline{n}\times\overline{n})^{<\infty}$, where $(a,a') \in T_{E,E'}$ if a and a' index partitions $C_1,...,C_n$ and $C_1',...,C_n'$ of some $\overline{k}$ in such a way that the C_i''s can each be extended to partitions of $\overline{k'}$ satisfying lines 3.20 - 3.21 for $C_1,...,C_n$. This tree is finitely branching and contains infinitely many branches, so there exists an infinite branch pair $(s,s') \in (\overline{n}\times\overline{n})^{\infty}$ with $(s,s')|_k \in T_{E,E'}$ for all k. Now let $D_1,...,D_n$ and $D_1',...,D_n'$ be two partitions of $\mathbb{N}$ with $m \in D_i$ iff $s_m = i$ and $m \in D_i'$ if $s_m' = i$. Since E,E' are both in $R_n^{\sim}$, all the D_i's and D_i''s must be infinite in size (otherwise E or E' is in $R_{n-1}^{\sim}$). Observe that for each i, $E_i := E|_{D_i}$ and $E_i' := E|_{D_i'}$ are isomorphically r.i. with distortion constant M. Finally, by definition of $T_{E,E'}$ and using induction, each of the $E_i's$ can be mapped injectively into ideals of E_i' using isomorphic embeddings with distortion at most M^2. By Corollary 3.4.12, E_i' is lattice isomorphic to E_i, so E is isomorphic to E'.

3.5 Non-Borel classes and relations of lattices

We consider certain classes of lattices that are not Borel. Many of the results here parallel the results in the Banach space setting, but a few are specific to lattices. For example, while the isomorphism relation for finite sums of rearrangement invariant isomorphically or isometrically invariant lattices is Borel, the general isomorphism relation on separable Banach lattices is not (and in fact, it is maximally complex for analytic equivalence relations, as will be demonstrated in Chapter 4). There are also certain isomorphism classes in particular which are analytic and not Borel. We can use these results to show that classes of lattices such as KB spaces and separable duals are co-analytic but not Borel and discuss some implications of these results.

We begin with the following:

Proposition 3.5.1. *The following relations are analytic:*

1. $Emb_\sim(X,Y)$: *X lattice isomorphically embeds into Y*

2. $Emb_\cong(X,Y)$: *X lattice isometrically embeds into Y*

3. $I_\sim(X,Y)$: *X is lattice isomorphic to Y*

4. $I_\cong(X,Y)$: *X is lattice isometric to Y*

Proof. Let $(\tau_n^m)_n$ enumerate all the m-ary functions for lattices generated by composing the operations of scalar multiplication over $\mathbb{Q}$, $+$, and $\wedge$. Define the following relation $\phi((x_i)_i,(y_i)_i,X,Y,M) \subseteq (\mathcal{U}^{\mathbb{N}})^2 \times BL^2 \times \mathbb{N}$ by

$$\phi((x_i)_i,(y_i)_i,X,Y,M) \iff (x_i)_i \subseteq X \bigwedge (y_i) \subseteq Y \tag{3.22}$$

$$\forall m,n \in \mathbb{N}\ \big(1/M\|\tau_n^m(x_1,...,x_m)\| \leq \|\tau_n^m(y_1,...,y_m)\| \leq M\|\tau_n^m(x_1,...,x_m)\|\big) \tag{3.23}$$

ϕ shows that the sequences $(x_i)_i$ and $(y_i)_i$ induce a lattice isomorphism with distortion at most M for the lattices generated by each sequence. Here the isomorphism is induced by $x_i \mapsto y_i$. By the equivalence in line 3.23, this map is well-defined and preserves lattice operations, and is an isomorphic embedding.

Now let $D \subseteq \mathcal{U}^{\mathbb{N}} \times BL$ be defined by

$$D((x_i),X) \iff (x_i)_i \subseteq X \bigwedge \forall m,n \in \mathbb{N}\ \exists k\ \|x_k - \psi_m(X)\| < 1/n,$$

D is the relation characterizing density of a sequence in X.

Both ϕ and D are Borel. From there, we easily obtain:

$$\begin{aligned}
Emb_\cong(X,Y) &\iff \exists (x_i)_i,(y_i)_i\ \phi((x_i)_i,(y_i)_i,X,Y,1) \bigwedge D((x_i)_i,X)\\
I_\cong(X,Y) &\iff \exists (x_i)_i,(y_i)_i\ \phi((x_i)_i,(y_i)_i,X,Y,1) \bigwedge D((x_i)_i,X) \bigwedge D((y_i)_i,Y)\\
Emb_\cong(X,Y) &\iff \exists (x_i)_i,(y_i)_i,M\ \phi((x_i)_i,(y_i)_i,X,Y,M) \bigwedge D((x_i)_i,X)\\
I_\sim(X,Y) &\iff \exists (x_i)_i,(y_i)_i,M\ \phi((x_i)_i,(y_i)_i,X,Y,M) \bigwedge D((x_i)_i,X) \bigwedge D((y_i)_i,Y)
\end{aligned}$$

All of these relations are clearly analytic. $\square$

Recall the definition in 1.2 of what it means to be Γ-hard or Γ-complete for a class Γ. Note that if we can show that some A is Σ_1^1-complete (Π_1^1-complete), we also prove that it cannot be Π_1^1 (Σ_1^1), and hence Borel. A common way to show that $B \subseteq Y$ is Σ_1^1- complete is to take some already known $A \subseteq X$ where X is Polish and A is Σ_1^1-complete, and find a continuous function $f : X \to Y$ where $f^{-1}(B) = A$. Note also that if B is Γ-hard (complete), $Y \backslash B$ is $\check{\Gamma}$-hard (complete).

The same comments stated above for regular Γ-hard (complete) sets can also be stated for Borel Γ-hard (complete sets). We can also show that a Σ_1^1 set A is Borel Σ_1^1-complete by finding a Borel function that maps a regular Σ_1^1-complete set to A.

We first show that the isomorphism relation on Banach lattices is not Borel by finding a Borel map f from some Polish space to $BL \times BL$ with $f(A) \subseteq I_\sim$ and $f(X \backslash A) \subseteq BL \times BL \backslash I_\sim$. The proof for this is patterned after the Bossard's proof that the isomorphism class of the Pelczynski universal Banach space is complete analytic (see [16]). The Pelczynski space is a Banach space with a universal basis that contains up to isomorphism any other separable space generated by a basic sequence as a complemented subspace. First, we construct a Lattice version of a universal Pelczynski space for spaces with unconditional basic sequences. For the following theorem and subsequent results, let $\mathcal{S} = \mathbb{N}^{<\mathbb{N}}$ and let the $\mathcal{N} = \mathbb{N}^\mathbb{N}$ equipped with the product topology on discrete $\mathbb{N}$. For $s \in \mathcal{S}$, let $|s|$ denote the length of s, and given two branches $s, t \in \mathcal{S}$, we denote $s \subseteq t$ if $|s| \leq |t|$ and for all $1 \leq i \leq |s|$, we have $s_i = t_i$. For $s \in \mathcal{S}$ or $s \in \mathcal{N}$, let $s|_n \in \mathcal{S}$ denote the tuple with the first n elements of s. If $t \in \mathcal{N}$ is an infinite branch, we also use the same notation $s \subseteq t$ if $s = t|_n$ for some n.

Theorem 3.5.2. *There exists a separable atomic order continuous lattice $\mathcal{V}$ with atoms (e_n) such that for any other separable order continuous atomic lattice A with generating atoms $(a_k)_k$ and for all $M > 1$, there is a subsequence $(e_{n_k})_k$ of $(e_n)_n$ such that the map $a_k \mapsto e_{n_k}$ induces an M-isometry between A and the band generated by $(e_{n_k})_k$. Furthermore, $\mathcal{V}$ is unique up to lattice isomorphism.*

Proof. This construction is similar to that found in Schechtman's' construction of the universal space for unconditional bases (see [69], as well as [54, pg 93]) We consider the universal separable lattice $\mathcal{U} = C(\Delta, \mathcal{L}_1(0,1))$. Let $(x_n)_n$ be dense in $\mathbf{S}(\mathcal{U})_+$. We define a bijection $\phi : \mathcal{S} \to \mathbb{N}$ such that if $t \subsetneq s$, then $\phi(t) < \phi(s)$. Given $b = (b_n)_n \in \mathcal{N}$, let sequences of the form $(\phi(b|_n))_n$ be called "suitable branches". Note that by the properties of ϕ, branches must be strictly increasing sequences in $\mathbb{N}$. Let $\mathcal{B}$ be the set of all suitable branches in $\mathcal{N}$.

Consider now the vector lattice $c_{00}(\mathbb{N})$ defined by finitely supported sequences in $\mathbb{R}^{\mathbb{N}}$ and equipped with the standard order. If $\beta = (b_n)_n \in \mathcal{B}$, we consider the characteristic function $\chi_\beta : \mathbb{N} \to \bar{2}$, with $\chi_\beta(j) = 1$ if j appears in the sequence β, and 0 otherwise. For $a = (a_n)_n \in c_{00}(\mathbb{N})$, define a norm $|||\cdot|||$ as follows:

$$|||a||| = \sup\{\left\|\sum_{j=1}^{n} \varepsilon_j \chi_\beta(j) a_j x_j\right\| : n \in \mathbb{N}, \varepsilon_j = \pm 1, \beta \in \mathcal{B}\}$$

The presence of the ε_j's are necessary to make $|||\cdot|||$ become a lattice norm. Now, let $e_n = (0, 0, ..., \overset{n}{1}, 0, 0, ...)$ denote the atoms in $c_{00}(\mathbb{N})$, and let $\mathcal{V} = \overline{c_{00}}^{|||\cdot|||}$, and note that $\mathcal{V}$ is an order continuous atomic lattice with atoms e_n forming a basis whose linear span is dense in $\mathcal{V}$.

Let A be an atomic order continuous lattice with $(u_k)_k$ as its atoms, and let $M > 1$ (we can also assume $M < 2$). By the universality of $\mathcal{U}$, we can assume $A \subseteq \mathcal{U}$. Since (x_n) is dense in $\mathcal{U}_+$, let $\varepsilon > 0$ and pick a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that for all $k \in \mathbb{N}$, $\|x_{n_k} - u_k\| < \frac{\varepsilon}{2^{k+2}}$.

From this it follows that as Banach spaces, A is isomorphic to the Banach space generated by $(x_{n_k})_k$, as its basic sequence, with the isomorphism generated by $u_k \mapsto x_{n_k}$. This isomorphism is M-isometric linearly for ε small enough: indeed, for all linear combinations $\sum r_k u_k$ with $\|\sum r_k u_k\| = 1$, we have

$$1 - \varepsilon/2 \leq |\|\sum r_k u_k\| - \|\sum r_k(u_k - x_{n_k})\| \tag{3.24}$$

$$\leq \|\sum r_k x_{n_k}\| \leq \|\sum r_k u_k\| + \|\sum r_k(u_k - x_{n_k})\| \leq 1 + \varepsilon/2 \tag{3.25}$$

Also, we use the following notation: if $s = (s_1, ..., s_k)$ and $n = \phi(s)$, let $v_n = x_{s_k}$. in other words, v maps n to the element indexed by the last element in $\phi^{-1}(n)$. Consider now the sequence $\beta = (b_k)_k$, where $b_k = \phi((n_1, ..., n_k))$. Observe that $\beta \in \mathcal{B}$ by definition of branches.

Now we want a subsequence of the atoms $(e_n)_n$ in $\mathcal{V}$ that is equivalent to v_{b_k}. It turns out the sequence (e_{b_k}) works. Note that since any linear combination is supported along a finite segment of β, we must have that

$$
\begin{aligned}
|||\sum_1^m c_j e_{b_j}||| = \|\sum_1^m |c_j| v_{b_j}\| = \|\sum_1^m |c_j| x_{n_j}\| \quad (3.26)\\
\sim_M \|\sum_1^m |c_j| u_j\| = \|\sum_1^m c_j u_j\|, \quad (3.27)
\end{aligned}
$$

since the u_j's are mutually disjoint. Hence A and the lattice generated by $(e_{b_k})_k$ are isomorphic as Banach spaces. The latter is an ideal, and hence a band, (since $\mathcal{V}$ is order continuous), generated by the atoms $(e_{b_k})_k$, and the map $u_k \mapsto e_{b_k}$ preserves disjointness and order, so it is a lattice isomorphism. Hence A is M-isometric to a band generated by a subsequence of $(e_n)_n$ in $\mathcal{V}$.

Now, we show that $\mathcal{V}$ is unique up to lattice isomorphism. This proof is an application of Pelczynski's decomposition method for unconditional bases tp the atoms in $\mathcal{V}$. Suppose $\mathcal{V}'$ is another atomic order continuous lattice sharing the same universal properties. Then there exist bands X and Y such that V is lattice isomorphic to a lattice sum of X and V' as bands, which will be denoted by $\mathcal{V} \sim \mathcal{V}' \oplus X$, and similarly $\mathcal{V}' \sim \mathcal{V} \oplus Y$. Now, since $\mathcal{V}$ is atomic order continuous, the lattice $(\oplus_{n\in\mathbb{N}}\mathcal{V})_2$ is also atomic and order continuous, which implies there is some Z such that $\mathcal{V} \sim (\oplus_{n\in\mathbb{N}}\mathcal{V}) \oplus Z$. From there, we have

$$\mathcal{V} \oplus \mathcal{V} \sim (\mathcal{V} \oplus_2 (\oplus_n \mathcal{V})_2) \oplus Z \sim (\oplus_n \mathcal{V})_2 \oplus Z \sim V$$

(In short, we have that $\mathcal{V} \sim \mathcal{V} \oplus \mathcal{V}$). We can argue similarly for $\mathcal{V}'$. Therefore,

$$\mathcal{V}' \sim \mathcal{V} \oplus Y \sim \mathcal{V} \oplus \mathcal{V} \oplus Y \sim \mathcal{V} \oplus \mathcal{V}' \sim \mathcal{V}' \oplus \mathcal{V}' \oplus X \sim \mathcal{V}' \oplus X \sim \mathcal{V}$$

Thus $\mathcal{V} \sim \mathcal{V}'$, and we are done. □

An example of a Σ^1_1-complete set is the set $IF \subseteq Tr$ (see [46, Theorem 27.1] for a proof). Here, Tr is the set of trees over $\mathbb{N}$, so we can think of any tree as a subset of the countable set $\mathcal{S}$. As a result, with an appropriate bijection between $\mathcal{S}$ and $\mathbb{N}$, Each possible branch of a tree can be encoded as a some (0,0,...,1,0,0,...) in Δ. A tree T, then, can be encoded as some element r in Δ, where $r_n = 1$ iff the branch encoded by r_n is in T. The set of trees in Δ is G_δ, (hence Polish). Given a tree T, let $[T]$ be the set of infinite branches of T. The set IF is that of **ill-founded trees**, that is, trees with an infinite branch). Observe then that $[\mathcal{S}] = \mathcal{N}$ and $T \in IF \iff \exists b \in \mathcal{N}(b \in [T])$, which describes an analytic relation.

Using $\mathcal{V}$, we will construct Banach lattices X_1 and X_2 and Borel maps $U_1 : Tr \to BL(X_1)$ and $U_2 : Tr \to BL(X_2)$ such that if $T \in IF$, then $U_i(T) \sim \mathcal{V}$, and if $U \in WF$, $U_1(T)$ has the Schur Property as a Banach space (meaning that any weakly null bounded sequence in X is norm null), and $U_2(T)$ is reflexive.

In order to do this, we replicate the technique found in [16] for proving the analytic completeness of the isomorphism class of the Pelczynski Banach space. We also introduce the following: an **interval** in the space $\mathcal{S}$ is a set of the form $\{t \in \mathcal{S} : s \subseteq t \subseteq s'\}$. Let $I = \{i_1, ..., i_k\}$ be a finite set of intervals in $\mathcal{S}$. We then say that I is an **admissible set of intervals** if every branch (finite or infinite) in $\mathcal{S}$ intersects at most one $i_j \in I$.

Now, we consider again $c_{00}(\mathcal{S})$, the vector lattice of finitely supported sequences in $\mathbb{R}^{\mathcal{S}}$ equipped with natural order, and indexed the elements by $\mathcal{S}$. Let $(u_s)_{s\in\mathcal{S}}$ be the set of standard unit vectors of $c_{00}(\mathcal{S})$, with u_s disjoint from u_t when $s \neq t$, and let $(e_n)_n$ be the sequence of generating atoms of $\mathcal{V}$. We construct two norms $|||\cdot|||_i$, where $i \in \{1, 2\}$, whose completions will give us nice spaces to work with.

For the norms, given $x = \sum_s x(s)u_s \in c_{00}(\mathcal{S})$, let

$$|||x|||_i = \sup\left\{ \left(\sum_{j=1}^{k} \left|\left| \sum_{s\in I_j} x(s)e_{|s|} \right|\right|^i \right)^{1/i} \ : \ \{I_1, ..., I_k\} \text{ admissable set of intervals} \right\}$$

Since $\|\cdot\|$ is a lattice norm, and the $e_{|s|}$'s are atoms in $\mathcal{V}$, $|||\cdot|||_i$ is also a lattice norm. Let X_i be the $|||\cdot|||_i$-closure on $c_{00}(\mathcal{S})$. Both X_1 and X_2 are two separable, atomic, order continuous lattices in BL, with $(u_s)_{s\in\mathcal{S}}$ as atoms whose linear span is dense in in each X_i. Now let A be a subset of $\mathcal{S}$. We can define $U_i(A)$ as the $|||\cdot|||_i$-closure of the lattice generated by the unit elements $\{u_s : s \in A\}$. Restricting U_1, U_2 to trees, we have just defined a map from Tr to $BL(X_i)$. We will show U_1 and U_2 are both Borel maps, but first, let us show how the maps work.

The following theorem and proof is identical in structure to the proofs found in [16], but we will include them here for completeness.

Theorem 3.5.3. *Let $\theta \in Tr$. If θ is ill-founded, then $U_i(\theta)$ is lattice isomorphic to $\mathcal{V}$. If θ is well-founded, then $U_1(\theta)$ has the Schur Property, and $U_2(\theta)$ is reflexive (hence* *neither can be isomorphic to $\mathcal{V}$).*

To prove this, we first use the following lemma:

Lemma 3.5.4. *Let $b \in \mathcal{N}$. Then the space $U_i(\{s \in \mathcal{S} : s \subseteq b\})$ is lattice isometric to $\mathcal{V}$. Furthermore, if $\theta \in Tr$ and b is an infinite branch in θ, then $U_i(\{s : s \subseteq b\})$ is a band in $U_i(\theta)$.*

Proof. Let $b \in \mathcal{N}$, and consider $b|_j = (b_1, ..., b_j)$. We show that the standard map $u_{b|_j} \mapsto e_j$ generates a lattice isometry. First, it preserves linear and lattice operations since atoms are being mapped to atoms. To show equality in norms, Let $y = \sum_{i=0}^{n} y(u_{b|_i}) u_{b|_i}$ with each $y(u_{b|_i}) \in \mathbb{R}$. Then we have

$$|||y|||_i = \left|\left|\left| \sum_{i=0}^{n} y(u_{b|_i}) u_{b|_i} \right|\right|\right|_i \tag{3.28}$$

$$= \sup \left\{ \left(\sum_{j=0}^{k} \left|\left| \sum_{s \in I_j} y(s) e_{|s|} \right|\right|^i \right)^{1/i} : \{I_j\} \text{ admissable interval} \right\} \tag{3.29}$$

$$= \sup \left\{ \left|\left| \sum_{s \in I} y(s) e_{|s|} \right|\right| : I \text{ interval in } b \right\} \tag{3.30}$$

$$= \left|\left| \sum_{i=1}^{n} y(b_{|i}) e_i \right|\right| \tag{3.31}$$

The above occurs because admissible intervals can contain at most one branch, so the supremum will only occur along a single interval, as well as the fact that the norm is a lattice norm. Finally, we note that if $b \subseteq A$, $U_i(A)$ is itself order continuous, and the atoms of $U_i(b)$ form an ideal in $U_i(A)$, and hence a band.

□

Lemma 3.5.5. *Let $(A_m)_{j \in \mathbb{N}}$ be a sequence of subsets of $\mathcal{S}$ such that any branch $b \in \mathcal{N}$ intersects at most one A_m. Then $U_i(\cup A_m)$ and $(\oplus_m U_i(A_m))_i$ (that is, the ℓ_i sum of the $U_i(A_m)$'s) are lattice isometric.*

The proof and notation are nearly identical to that [16, Lemma 1.5], and are included here for completion:

Proof. Suppose $x \in U_i(\cup_j A_j)$ with finite support, and $x_m = \sum_{s \in A_m} x(s) e_s$. Then for some $M \in \mathbb{N}$, we have $x = \sum_1^M x_m$. Let $(I_j)_{j=1}^k$ be an admissible choice of intervals, let

$I_j(x) = \sum_{s\in I_j} x(s)e_s$, and let $M_m = \{j \in \mathbb{N} : I_j \cap A_m \neq \emptyset\}$, that is, the set of indices for the I_j's intersecting A_m. Observe that the M_m's form an M-partition $\{1, ..., k\}$ Finally, let J_j^m be the largest interval whose ends are in $I_j \cap A_m$. Now for each m, $\{J_j^m : j \in M_m\}$ is a choice of admissible intervals, since they are sub-intervals of some of the intervals in (I_j). Also, for each j, $I_J(x) = J_j^m(x) = J_j^m(x_m)$ for some m, so we then have

$$\sum_{j=1}^{k} \|I_j(x)\|^i = \sum_{m=1}^{M} \sum_{j\in M_m} \|J_j^m(x_m)\|^i \leq \sum_{m=1}^{M} |||x_m|||_i^i.$$

so $|||x|||_i^i \leq \sum_{m=1}^M |||x_m|||_i^i$. To show the opposite, for $1 \leq m \leq M$, let $(I_j^m)_{j=1}^{k_m}$ be an admissible choice of intervals, and let J_j^m be the largest interval in I_j^m with its ends in $I_j^m \cap A_m$. Because of the initial assumption on the A_m's, we have that $\{J_j^m : j \leq k_m, m \leq M\}$ is an admissible choice of intervals. Furthermore, for all m and j, we have $I_j^m(x_m) = J_j^m(x_m) = J_j^m(x)$, so

$$\sum_{m=1}^{M} \left(\sum_{j=1}^{k_m} \|I_j^m(x_m)\|^i \right) = \sum_{m=1}^{M} \sum_{j=1}^{k_m} \|J_j^m(x)\|^i \leq |||x|||^i,$$

So we also have $\sum_1^M |||x_m|||_i^i \leq |||x|||_i^i$. Then the natural map from $U_i(\cup A_m) \to (\oplus_m U_i(A_m))_i$ generated by sending any atom $e_s \in U_i(\cup A_m)$ with $s \in A_m$ for some m, to the atom e_s in $U_i(A_m) \subseteq (\oplus_m U_i(A_m))_i$. is a lattice isometry. □

Finally, we just state the following without proof:

Lemma 3.5.6. *(Bossard) Let (X_j) be a sequence of Banach spaces with the Schur Property. Then the resulting lattice $(\oplus X_j)_1$ has the Schur Property.*

Now that we have the necessary lemmas, we can finally prove the theorem:

Proof of Theorem 3.5.3. Let Θ be ill-founded, and suppose $b \in [\Theta]$. Now, note that the atoms in $U_i(s : s \subseteq b)$ generate the lattice itself, and by Lemma 3.5.4, $\mathcal{V}$ embeds isometrically into $U_i(\Theta)$, so $U_i(\Theta)$ satisfies the universality and order continuity properties that define $\mathcal{V}$. But by uniqueness of $\mathcal{V}$, $\mathcal{V}$ is lattice isomorphic to $U_i(\Theta)$.

Now suppose Θ is well-founded. We will use the same notation as in [16, Theorem 1.2]: Let $s, t \in \mathcal{S}$ and suppose $A \subseteq \mathcal{S}$. If $s = (s_1, ..., s_m)$ and $t = (t_1, ..., t_n)$, let $s^\frown t = (s_1, ..., s_m, t_1, .., t_n)$. We also define the following sets:

$$s^\frown A = \{s^\frown t : t \in A\} \qquad\qquad A_i = \{t \in \mathcal{S} : (i)^\frown t \in A\}$$

We will give a rank-type argument to show the following: $U_1(s^\frown\Theta)$ has the Schur Property, and $U_2(s^\frown\Theta)$ is reflexive. Let $\rho(\Theta) = \alpha < \omega_1$, and suppose that for all well founded trees T with $\rho(T) < \alpha$, and for all $s \in \mathcal{S}$, we have $U_1(s^\frown T)$ with the Schur property and $U_2(s^\frown T)$ reflexive. Let $s \in \mathcal{S}$. Let $N_s = \{i \in \mathbb{N} : s^\frown(i) \in \Theta\}$, and for $j \in N_s$, let $A_j = s^\frown((j)^\frown\Theta_j)$. In other words, A_j consists of s attached to the branches in Θ that begin with j. Then we have $\bigcup_{i\in N_s} A_j = (s^\frown\Theta)\backslash\{s\}$. Note also that each branch $b \in \mathcal{N}$ meets at most one of these A_j's, so by Lemma 3.5.5,

$$U_i\big((s^\frown\Theta)\backslash\{s\}\big) = U_i\big(\bigcup_{j\in N_s} A_j\big) = (\oplus_{j\in N_s} U_i(A_j))_i$$

Hence $U_2\big((s^\frown\Theta)\backslash\{s\}\big)$ is reflexive (it is a 2-sum of reflexive spaces) and by Lemma 3.5.6, $U_1\big((s^\frown\Theta)\backslash\{s\}\big)$ has the Schur Property. To get prove the same for $s^\frown\Theta$, we note that the set $\{u_{s^\frown t} : t \neq \emptyset, t \in \Theta\} \subseteq X_i$ forms a basis of atoms generating $U_i\big((s^\frown\Theta)\backslash\{s\}\big)$, so each lattice $U_i(s^\frown\Theta)$ is generated by adding the atom u_s to the generating basis. Hence $U_i(s^\frown\Theta)$ is lattice isomorphic to $\mathbb{R} \oplus U_i\big((s^\frown\Theta)\backslash\{s\}\big)$, which implies the needed fact. From there, if we let $s = \emptyset$, we have shown that $U_1(\Theta)$ has the Schur Property, and $U_2(\Theta)$ is reflexive. $\square$

Each U_i sends ill-founded trees to lattices isomorphic to copies of $\mathcal{V}$, while well-founded trees are sent to lattices clearly non-isomorphic to $\mathcal{V}$. It remains to that each U_i is a Borel map.

Lemma 3.5.7. *The maps $U_i : Tr \to BL(X_i)$ are Borel.*

Proof. We show that the pre-image of a Borel set is Borel. Recall that the Borel $\sigma-$algebra is generated by sets of the form $V_U = \{F \in BL(X_i) : F \cap U \neq \emptyset\}$, where $U \subseteq X_i$ is open. Pick U open, and consider the preimage $U_i^{-1}(V_U)$. Note then that for $\Theta \in Tr$, $U_i(\Theta) \in V_U$ iff $U_i(\Theta) \cap U \neq \emptyset$ iff there exists $k \in \mathbb{N}, s_1, ..., s_k \in \Theta, \lambda_1, .., \lambda_k \in \mathbb{Q}^k$ such that $\sum_{i=1}^k \lambda_i u_{s_i} \in U$. Then we have

$$\Theta \in U_i^{-1}(V_u) \iff \exists k \in \mathbb{N}, \lambda \in \mathbb{Q}^k, s \in \mathcal{S}^k \left(\bigwedge_{i=1}^{|s|} s_i \in \Theta \bigwedge \sum_i \lambda_i u_{s_i} \in U \right)$$

The quantification is over countable sets, and the relation in the parenthesis is a clopen subset of $\mathcal{S}^{<\infty} \times \mathbb{Q}^{<\infty} \times Tr$. Hence U_i is Borel.

□

Putting it all together, we have the following:

Theorem 3.5.8. *The isomorphism class $< \mathcal{V} >$, and thus the lattice isomorphism relation $I_\sim$, as a set is analytic non-Borel.*

Here we consider $I_\sim$'s descriptive set theoretic complexity. The complexity of $I_\sim$ as an equivalence relation is addressed in Chapter 4.

Proof. We consider the Borel map $U_1 : Tr \to BL(X_1)$. Note that X_1 can be embedded isometrically into the universal separable lattice $\mathcal{U}$, so we can just consider $X_1 \subseteq \mathcal{U}$. Furthermore, $BL(X_1) \subseteq BL$ is Borel. So we can simply consider the map U_1 to take trees to lattices in BL. Now, the isomorphism relation is analytic, and so the isomorphism class $< \mathcal{V} >$ is analytic as well. From Lemma 3.5.7, U_1 is a Borel map that separates well founded trees from ill-founded trees, with $U_1^{-1}(< \mathcal{V} >) = IF$. Hence $< \mathcal{V} >$ is complete analytic. To show the isomorphism equivalence relation is itself analytic, we just consider the map $f : Tr \to BL \times BL, f(\Theta) \mapsto (U_1(\Theta), \mathcal{V})$. This too is Borel, and $f^{-1}(I_\sim) = IF$. Hence $I_\sim$ is complete analytic.

□

We can use these maps to show the non-Borelness of other classes of lattices. First, we show that the additional lattice structure does not simplify the descriptive complexity of certain classes of Banach spaces:

Corollary 3.5.9. *The following classes of Banach lattices are complete co-analytic:*

1. *Reflexive lattices*
2. *Lattices with separable dual*
3. *Lattices not containing an arbitrary infinite dimensional order continuous atomic lattice Z.*

Proof. The case for reflexive lattices can be proven as in with Banach spaces (see [16, Corollary 3.3], for instance), but here we provide a characterization specific to lattices: By [56, Proposition 3.1], a lattice X is reflexive iff it does not contain isomorphic copies of c_0 or ℓ_1; i.e., if $\neg Emb_{\sim}(c_0, X) \bigwedge \neg Emb_{\sim}(\ell_1, X)$. By Proposition 3.5.1, this relation is co-analytic. For completeness, use the map U_2: if Θ is well-founded $U_2(\Theta)$ is reflexive, if not, then $U_2(\Theta)$ is isomorphic to $\mathcal{V}$, which is isomorphically universal and thus not reflexive.

The co-analycity of lattices with separable duals can be shown using the proof in [16, Corollary 3.3], and completeness is also demonstrated with U_2.

Finally, for (3), we consider two cases: if Z is not reflexive, use U_2 to map well-founded trees to reflexive lattices and ill-founded trees to $\mathcal{V}$. If Z is reflexive, use U_1 to map well-founded trees to lattices with the Schur Property (which cannot contain infinite dimensional reflexive lattices), and ill-founded trees to $\mathcal{V}$, which contains a copy of Z. Clearly X does not contain Z isomorphically if $\neg Emb_{\sim}(Z, X)$, so the relation is co-analytic as well.

$\square$

We can also demonstrate the descriptive complexity of classes are specific to the Lattice setting:

Corollary 3.5.10. *The following classes of Banach lattices are complete co-analytic:*

1. *KB spaces*
2. *Spaces with the Radon-Nikodým Property (RNP)*
3. *Spaces with the Krein-Milman Property (KMP)*
4. *Spaces with the Solid Krein-Milman Property (SKMP)*
5. *Dual spaces.*

Note: Bossard also proved complete co-analycity for Banach spaces with the RNP. We give an alternate proof here, using properties specific to Banach Lattices.

Proof. We first note each of these classes are co-analytic, since the relation can easily be defined as such.

(1): For KB- spaces, if $X \in BL$ we have

$$X \in KB \iff \forall (x_i) \in X_+^{\mathbb{N}}, \bigwedge_i 0 \leq x_i \leq x_{i+1} \wedge \|x_i\| \leq 1 \implies (x_i) \text{ is Cauchy}$$

(2)-(5) : These are equivalent for separable Banach lattices. (2) is equivalent to (3) in Banach lattices (for a short proof see [20]). Theorem 2.6.1 proves the equivalence of (3) and (4), and a result from Talagrand (Corollary 5.4.21 in [61]) shows the equivalence between (3) and (5). To prove co-analycity, we use another result by Talagrand (Corollary 5.4.20 in [61]), which states that a Banach lattice has the RNP iff it is an order dentable KB space. Recall that X is *order dentable* if for every closed convex subset C of X_+, if $C \subseteq I_X(e)$ for some $e \in X_+$, then

$$C \neq \bigcap_n \overline{CH}(\{y \in C : \|y \wedge e\| \leq \frac{1}{n}\}),$$

where $\overline{CH}$ denotes the closed convex hull of a set. It is easy to show that the map $C \mapsto \{y \in C : \|y \wedge e\| \leq \frac{1}{n}\}$ is Borel. By Proposition 2.2.9, the map from (C, n, e) to the set $D(C, n, e) := \overline{CH}(\{y \in C : \|y \wedge e\| \leq \frac{1}{n}\}$ is Borel. Now note then that for all n, if C is convex, $D(C, n, e) \subseteq C$. Let OD be the class of order dentable lattices; then we can define order dentability as follows:

$$OD(B) \iff \forall C \in F(B)\ \forall e \in B_+ \tag{3.32}$$
$$\big((C = \overline{CH}(C) \bigwedge C \subseteq I_B(e)) \implies \exists n D(C, n, e) \neq C\big) \tag{3.33}$$

The class of order dentable sets is thus a co-analytic class. the intersection of co-analytic sets is co-analytic, so $OD \cap KB$ remains co-analytic.

Now we use the map U_2, noting that if $T \in WF$, $U_2(T)$ is reflexive, and hence KB and a dual space, so it fulfills all classes (1)-(5). However, if $T \in IF$, then $U_2(T) \sim \mathcal{V}$, and thus cannot be KB, since it contains a lattice copy of c_0. Thus $U_2(T)$ is not in any of the classes listed above. □

Corollary 3.5.10 implies the following negative universality result:

Corollary 3.5.11. *There is no separable KB lattice that is isometrically or isomorphically universal for separable KB lattices.*

Proof. We prove the above for the isomorphic case, since the isometric case is identical. Suppose W is such a lattice. The property of being KB is closed under sublattices. Thus it follows that any separable X is KB if it lattice isomorphically embeds into W, that is, if $Emb_{\sim}(X, W)$. But the latter is an analytic class by Proposition 3.5.1, so since KB is co-analytic, it also is Borel by Souslin's Theorem. But this contradicts Corollary 3.5.10, so no such W exists. □

Remark 3.5.12. The approach used here can also be applied to any isomorphically or isometrically hereditary property. That is, if C is a non-Borel co-analytic class of lattices, and C is also isomorphically or isometrically closed under sublattices, then there is no X in C that is isomorphically or isometrically universal for all lattices in C.

3.6 Questions and further research

Question 3.6.1 (Complexity of Fatouness)**.** The property of having a (weak) Fatou norm is used to characterize "band-like" behavior, and thus plays a role in characterizing general atomicity, as opposed to individual atoms. What is the complexity of the Fatou Property in a Banach lattice? Writing down the brute definition of having a Fatou norm involves $\forall\exists$, making it at most a Π^2_1 relation, but equivalent characterizations tend to be elusive.

Question 3.6.2 (Universal lattices)**.** Theorem 3.5.2 demonstrates the existence of an isomorphically universal order continuous atomic separable lattice, while Corollary 3.5.11 shows that no such universal lattice exists for the class of KB spaces. For what classes C of separable lattices is there a lattice in C that also isomorphically or isometrically contains all lattices in C? Remark 3.5.12 gives an approach for arriving at negative results for co-analytic non-Borel classes that are closed under subspaces, but other methods are needed when a class is not complete co-analytic or hereditarily closed. For instance, is there an order continuous separable lattice that is universal for separable lattices? Or, is there a universal lattice that has a Fatou norm?

Question 3.6.3 (Atomicity)**.** Corresponding to Fatouness is general atomicity. Theorem 3.3.2 shows that atomic lattices form a co-analytic class, but whether it is Borel is an open question. Attempts have been made to prove non-Borelness in an AM setting, only to realize that by Theorem 3.3.4, the atomic $C_0(K)$ spaces form an analytic set as well. To prove non-Borelness, we would have to leave the world of Fatou norms.

Question 3.6.4 (Rearrangement invariance in order continuous lattices)**.** Rearrangement invariance is not limited to atomic lattices. Recall the definition of a Köthe function space from [55, Definition 1.b.17]: Let (Ω, Σ, μ) be a complete, σ-finite measure space. Then a Banach space X is called a **Köthe function space** if

1. X is closed, and its norm is preserved, under downward order. That is, if $x \in X$ and $|y| \leq |x|$ a.e., then $y \in X$ and $\|y\| \leq \|x\|$.

2. For all measurable sets $F \in \Sigma$, $\chi_F \in X$.

Köthe function spaces are clearly lattices. In addition, by [55, Theorem 1.b.14], Any order continuous separable lattice X can be represented as a Köthe function space over some complete, σ-finite (Ω, Σ, μ) such that $L_\infty(\Omega) \subseteq X \subseteq L_1(\Omega)$.

Based on this, we say an order continuous Köthe function space X is then **rearrangement invariant** if any measure preserving transformation on Ω induces an isometry on X. What is the complexity of rearrangement invariance for order continuous separable lattices? When the lattices are atomic, "measure preserving transformations" on $\mathbb{N}$ are simply the permutations in S_∞, assuming the singleton sets in $\mathbb{N}$ have measure 1. For non-atomic or atomless lattices, one might need to find a way to discern the underlying measure space. A similar issue arises in the following chapter in which a Borel map is needed to deduce the underling K in lattices isometric for $C(K)$ spaces.

Chapter 4

Complexity of equivalence relations

In this chapter, we consider specifically the complexity of equivalence relations on $BL(X)$ (or just BL), particularly, the lattice isomorphism and lattice isometry relations. For this, we need the following definitions:

Definition 4.0.1. *Let X, Y be two Polish spaces, and let E, F be equivalence relations on X, Y respectively. A map $f : X \to Y$ is called a* ***homomorphism*** *if for all $x, y \in X$,*

$$xEy \implies f(x)Ff(y)$$

If in addition, f is Borel, and in addition

$$xEy \iff f(x)Ff(y)$$

the map is called a ***Borel reduction****. We say then that E is* ***Borel reducible*** *to F, and denote it by $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, then E is* ***bireducible to F****, and denote $E \sim F$.*

We also consider some examples of equivalence relations that may appear later in my research:

- The identity relation $id(X)$: given a Polish space X, with $x, y \in X$, we have $x \sim y \iff x = y$. By the Borel isomorphism theorem, for all X, Y, we have not just reducibility but an isomorphism (that is, a Borel isomorphism preserving the equivalence) between equivalence relations. More specifically, we say that an equivalence relation E on X is **concretely classifiable** or **smooth** if $E \leq_B id(Y)$ for some Polish Y. Note that since any two Polish spaces are Borel isomorphic iff the have the same cardinality (see [46, Theorem 15.6]), we commonly let $Y = \Delta$, where Δ is the cantor set, if we want to show smoothness.

- E_0: a relation on Δ, with $xE_0y \iff \exists n \forall m \geq n(x_m = y_n)$. This relation is bireducible to the relation on $\mathbb{R}$ given by $x \sim y \iff x - y \in \mathbb{Q}$. A result from Harrington, Kechris, and Louveau (see [42]) gives a dichotomy between $id(\Delta)$ and E_0 for Borel equivalence relations, stating that for any such Borel relation F on a space X, you have either $F \leq_B id(\Delta)$, or $E_0 \sqsubseteq_c F$ (i.e., there exists a continuous injective reduction from F to E_0)

- The logic actions: Let $L = (R_i)$ be a countable relational language, with R_i having arity n_i. Suppose you have a countable structure $\mathbf{M}$ with underlying set $\mathbb{N}$. Then $\mathbf{M}$ can be represented as an element x in

  $$X_L := \prod 2^{\mathbb{N}^{n_i}}$$

  Where $R_i^M(k_1, ...k_{n_i}) \iff x_{(k_1,...,k_{n_i})} = 1$. Let $Mod(\mathcal{L})$ be the collection of countable $\mathcal{L}$-structures, encoded by X_L. Two structures M, N are isomorphic if there exists a bijection $g : \mathbb{N} \to \mathbb{N}$ such that $\forall i \in \mathbb{N}, (k_1, ..., k_{n_i}) \in \mathbb{N}^{n_i}$, we have

  $$(*) \qquad R_i^M(g(k_1), ..., g(k_{n_i})) \iff R_i^N(k_1, ..., k_{n_i})$$

  Note that this isomorphism is a permutation of $\mathbb{N}$, and thus an element of S_∞, itself a Polish space. Conversely, since $Mod(\mathcal{L})$ also ranges over all possible countable L-structures, a permutation g of the underlying set $\mathbb{N}$ of $\mathbf{M}$ can induce another structure $\mathbf{N} \in Mod(L)$ where the interpretation is defined in terms of the interpretation on $\mathbf{M}$ exactly as written in $(*)$. In short, we have defined a group action ofS_∞ on $Mod(\mathcal{L})$. We can then define an equivalence relation $\cong_L$ on $Mod(\mathcal{L})$ by

  $$M \cong_L N \iff \exists g \in S_\infty \text{ such that } g \cdot \mathbf{M} = \mathbf{N}$$

  The equivalence classes are simply the S_∞ orbits. We say that a relation E is **classifiable by countable structures** if it is Borel reducible to $\cong_L$ for some countable language $\mathcal{L}$.

- More generally, consider a Polish group G acting on a Polish space X, then the action of G on X induces an equivalence relation, where the equivalence classes of $x \in X$ is the G-orbit of X. In fact, all the above can be expressed as a Polish group action on a Polish space. However, Kechris and Louveau proved that there exist equivalence relations that are not reducible to Polish group orbit relations. The simplest such example is E_1, defined on $\mathbb{R}^{\mathbb{N}}$, with

$$(x_n) \sim (y_n) \iff \exists n \forall m \geq n(x_n = y_n).$$

4.1 The lattice isometry equivalence relation

Let's look first the isometry case. In this section, we show that the lattice isometry equivalence relation $I_{\cong}$ on separable Banach lattices is Borel bi-reducible to the universal relation induced by a Polish group over a standard Borel space. We first show that it is reducible to such a group: the approach is an application of concepts and results found in [30], in which the complexity of isometry and isomorphism relations for C^*-algebras are shown to be under the action of a Polish group.

The following definition is taken from [30, Definition 2.1]:

Definition 4.1.1. *Let $\mathcal{L} = (l_1, l_2, ...)$ be a finite or infinite sequence in $\mathbb{N}$. Then a **Polish $\mathcal{L}$-structure** is a triple*

$$\mathbb{X} = (X, d^{\mathbb{X}}, (F_n^{\mathbb{X}})_n)$$

*Where $(X, d^{\mathbb{X}})$ is a Polish space equipped with complete metric $d^{\mathbb{X}}$ and for each n, $F_n^{\mathbb{X}} \subseteq X^{l_n}$ is a closed subset in the product topology. Here, $\mathcal{L}$ is called the **signature** of $\mathbb{X}$ and $(X, d^{\mathbb{X}})$ its **domain**.*

For instance, if X is a separable lattice, we can consider the Lattice operations on X represented by symbols $+, \vee, \wedge, q\cdot$ to get the Polish structure induced by $F_+, F_\vee, F_\wedge, (F_q)_{q\in\ Q}$, the graphs of functions of vector addition, join, meet, and multiplication by q for each $q \in \mathbb{Q}$. Then the following structure

$$\mathbb{X} = (X, d^{\mathbb{X}}, F_\vee^{\mathbb{X}}, F_\wedge^{\mathbb{X}}, F_+^{\mathbb{X}}, (F_q^{\mathbb{X}})_{q\in\mathbb{Q}})$$

is an instance of a Polish structure with signature $(2, 2, 2, 3, 3, 3, 3, ...)$.

Given a Polish space (Y, d), $\mathcal{L} = (l_1, l_2, ...)$ the signature for a countable language, and consider the set $\mathcal{M}(\mathcal{L}, Y, d) \subseteq (\mathcal{F}(Y)\backslash\emptyset) \times \Pi_n \mathcal{F}(Y^{l_n})$

$$\mathcal{M}(\mathcal{L}, Y, d) = \{X, (F_n) : F_n \subseteq X^{l_n}\}$$

Note that $\mathcal{M}(\mathcal{L}, Y, d)$ is Borel. If we let Y be the Urysohn space $\mathbb{U}$, then simply denote $\mathcal{M}(\mathcal{L}, \mathbb{U}, d)$ by $\mathcal{M}(\mathcal{L})$. Note then that this characterizes Polish structures over the language $\mathcal{L}$. Is it possible then, to characterize Banach lattices as a Borel subset of $\mathcal{M}(\mathcal{L})$? [30, Lemma 3.1] gives the affirmative:

Lemma 4.1.2. *(Elliot, et. al.) If T is a theory in a countable language in the logic of metric structures then the set of all $\mathbb{X} \in \mathcal{M}(\mathcal{L})$ encoding a model of T is Borel.*

It is already known that the theory of Banach Lattices can be axiomatized using continuous logic. One can, for instance, take the Banach lattice axioms outlined in [68, Chapter II, Definitions 1.1-1.2, 5.1], and write down an axiomatization that completely characterizes Banach lattices using the signature $(+, -, 0, \mathbb{R}, \wedge, \vee, \|\cdot\|)$. Furthermore, any axioms $r \in \mathbb{R}$ can be reduced to axioms only involving $r \in \mathbb{Q}$, so one needs only countably many axioms with countably many function and predicate symbols to characterize lattices. Thus Lemma 4.1.2 applies.

So we can also consider BL to be a Borel subset of $\mathcal{M}(\mathcal{L})$, where the unit ball of $\mathcal{L}$ is a language whose signature consists of symbols corresponding to the graphs of averaged addition $+$, modulus $\|\cdot\|$, and scalar multiplication $\cdot q$ by a rational q with $|q| \leq 1$. Now, suppose that $\mathbb{X}, \mathbb{Y}$ are two Polish structures with the same signature. Then $\mathbb{X}, \mathbb{Y}$ are said to be **isometrically isomorphic as Polish structures** if there exists an isometric bijection $h : X \to Y$ on the underlying sets such that we have

$$(x_1, ..., x_{l_n}) \in f_n^{\mathbb{X}} \iff (h(x_1), ..., h(x_{l_n})) \in f_n^{\mathbb{Y}}.$$

and is denoted by $\mathbb{X} \simeq \mathbb{Y}$. If $\mathcal{L}$ is a signature for Banach lattices, an isometric isomorphism between structures $\mathbb{X}, \mathbb{Y}$ in BL corresponds to a lattice isometry between X and Y.

Consider now the isometries on $\mathbb{U}$, denoted by $Iso(\mathbb{U})$. An isometry over $\mathbb{U}$ can be extended to Polish structures as well: given a structure $\mathbb{X} = (X, f_n^{\mathbb{X}}) \in \mathbb{M}(\mathcal{L})$ and $\sigma \in Iso(\mathbb{U})$, we let $\sigma \cdot \mathbb{X} = (\sigma(X), \sigma(f_n^{\mathbb{X}}))$, with σ acting component-wise on each relation $f_n^{\mathbb{X}}$. This induces another equivalence relation, this time on $\mathbb{U}$, denoted by $\equiv$, where

$$\mathbb{X} \equiv \mathbb{Y} \iff \exists \sigma \in Iso(\mathbb{U})(\sigma \cdot \mathbb{X} = \mathbb{Y})$$

Note that this equivalence relation is another equivalence relation of a Polish group on a standard Borel space. From a direct application of [30, Theorem 3.2], we thus have the following:

Theorem 4.1.3. *$I_{\cong} \leq_B \equiv$, so $I_{\cong}$ is below a Polish group action induced by $Iso(\mathbb{U})$ on a standard Borel space.*

Given that Banach lattices can be axiomatized, we can then restrict the Borel reduction from the above theorem to models of the theory of Banach Lattices and conclude that the lattice isometry relation is Borel reducible to $Iso(\mathbb{U})$ acting on $(F(\mathbb{U})\backslash\emptyset) \times \prod \mathbb{U}^{l_n}$.

What about a lower bound? It turns out that in fact, isometry on Banach lattices is bireducible with the universal equivalence relation of Borel actions by a Polish group. We use the main result from Zielinski in [72], which states that the universal relation for Polish group actions on standard Borel spaces is bireducible with the equivalence relation of homeomorphisms of compact metrizable spaces.

By the Banach-Stone theorem, two compact metric spaces K and K' are homeomorphic iff the corresponding Banach lattices $C(K), C(K')$ are isometric. In addition, if $\phi : K \to K'$ is a surjective continuous map, then $C(K')$ embeds lattice isometrically into $C(K)$ via the map $f \mapsto f \circ \phi$. Thus if we construct a map $K \mapsto C(K)$ in a Borel way, this will prove the Borel reduction. It is apparently known that such a mapping apparent (see the comments in [39, Problem 10.2]), but here we will give an explicit construction.

First, we need suitable ambient spaces for K and $C(K)$. It is already known that K is homeomorphic to a closed subspace of the Hilbert cube $\mathbb{H}$, so we just consider $K \subseteq \mathbb{H}$. Furthermore, we have that $\Delta \twoheadrightarrow K$ For any compact $K \subseteq \mathbb{H}$. Hence isometric copies of $C(K)$'s sit in $C(\Delta)$. We construct a Borel map $C : \mathcal{K}(\mathbb{H}) \to BL(C(\Delta))$, with $K \mapsto C(K)$ over various steps as follows:

Lemma 4.1.4. *There exists a surjection ϕ from Δ to $\mathbb{H}$ such that the map*

$$K \mapsto T_K := \phi^{-1}(K) \subseteq \Delta$$

is Borel.

Proof. For simplicity's sake, we assume the metric on $\mathbb{H}$ is $d(x, y) = \sum_i 2^{-i}|x_i - y_i|$, and the metric on Δ is $d(x, y) = \sup_{x_k \neq y_k} 2^{-k}$. Let $S = 2^{<\mathbb{N}}$, and for $s \in S$, let N_s

consist of the elements in Δ that begin with s. We construct a collection of closed sets $(F_s)_{s\in S}$ such that:

- $F_\emptyset = \mathbb{H}$
- $F_{s^\frown 0} \cup F_{s^\frown 1} = F_s$
- F_s is non-empty for all s.
- $\mathrm{diam}(F_s) \leq \delta(|s|)$, where δ is a decreasing function on $\mathbb{N}$ with $\delta(n) \to 0$ as $n \to \infty$

Such a construction induces a continuous surjective map ϕ, with $\phi(N_s) = F_s$. the map takes a sequence $x \in \Delta$ to the unique $y \in \mathbb{H}$ such that $\{y\} = \bigcap_n F_{x|_n}$. This implies the map $K \mapsto T_K$ is Borel, since the graph is defined by

$$G(K,T) \iff \forall s(T \cap N_s \neq \emptyset \iff K \cap F_s \neq \emptyset).$$

Indeed, suppose the right hand side of above is true for $T \subseteq \Delta$. If $x \in T$, then for all $n, T \cap N_{x|_n} \neq \emptyset$, hence $K \cap F_{x|_n} \neq \emptyset$ which implies that $\phi(x) \in K$. Hence $T \subseteq T_k$. similarly, if $\phi(x) \in K$, then $K \cap F_{x|_n} \neq 0$ for all n, hence $N_{x|_n} \cap T \neq \emptyset$, implying that $x \in T$, so $T_K = T$.

The condition $T \cap N_s \neq \emptyset$ is clopen in $\mathcal{K}(\Delta)$ and $K \cap F_s \neq \emptyset$ closed in $\mathcal{K}(\mathbb{H})$, so the map is G_δ, hence Borel.

Finally, we sketch the construction of F_s. Let $Q_i^j = [2^{-i}j, 2^{-i}(j+1)] \subseteq [0,1]$, where $i \in \mathbb{N}$ and $0 \leq j < 2^i$. Let $F_0 = Q_1^0 \times \mathbb{H}$, and $F_1 = Q_1^1 \times \mathbb{H}$. Note then that $\mathrm{diam}(F_\emptyset) = 2$, but $\mathrm{diam}(F_i) = 1$. For $k = 0, 1$, let

$$F_{(i,k)} = Q_1^i \times Q_1^k \times \mathbb{H},$$

and let

$$F_{(i,k,l)} = Q_2^{2i+l} \times Q_1^k \times \mathbb{H}.$$

Subsequent closed sets F_s are of the form

$$F_s = Q_m^{i_0} \times \ldots \times Q_m^{i_{m-1}} \times Q_k^{i_m} \times \mathbb{H},$$

with $k \leq m$, or of the form

$$F_s = Q_m^{i_0} \times ... \times Q_m^{i_k} \times Q_{m-1}^{i_{k+1}} \times ... \times Q_{m-1}^{i_{m-2}} \times Q_{m-1}^{i_{m-1}} \times \mathbb{H}.$$

If F_s is of the first form and $k < m$, then for each $l = 0, 1$, let

$$F_{s^\frown l} = Q_m^{i_0} \times ... \times Q_m^{i_{m-1}} \times Q_{k+1}^{2i_m+l} \times \mathbb{H}.$$

Otherwise, if $k = m$, let

$$F_{s^\frown l} = Q_{m+1}^{2i_0+l} \times Q_m^{i_1} \times ... \times Q_m^{i_m} \times \mathbb{H}.$$

If F_s is of the second form and $k < m - 1$, then let

$$F_{s^\frown l} = Q_m^{i_0} \times ... \times Q_m^{i_k} \times Q_m^{2i_{k+1}+l} \times ... \times Q_{m-1}^{i_{m-1}} \times Q_{m-1}^{i_m} \times \mathbb{H}.$$

Otherwise, if $k = m - 1$ let

$$F_{s^\frown l} = Q_m^{i_0} \times ... \times Q_m^{i_{m-1}} \times Q_1^l \times \mathbb{H},$$

Clearly for each s, $F_{s^\frown 0} \cup F_{s^\frown 1} = F_s$, and for s with $|s| \geq N$, we have that $\text{diam}(F_s) \leq 2^{2-\sqrt{|s|}}$, satisfying the necessary conditions for $(F_s)_{s \in S}$. $\square$

Now, for any $K \in \mathbb{H}$, we have $T_K \in \mathcal{K}(\Delta)$. But any closed subset T of Δ is a retract. That is, there exists a continuous map $\psi : \Delta \to T$ such that $\psi|_T = id_T$. This implies that any K is a continuous surjective image of Δ. Furthermore, this ψ_T can be chosen in a "nice" way in relation to T.

Lemma 4.1.5. *There is exists a continuous map $\psi : \mathcal{K}(\Delta) \to C(\Delta, \Delta)$ sending T to a retract $\psi_T : \Delta \to \Delta$ of Δ to T.*

Proof. First, we show that the set of retractive maps R is closed: Suppose $\psi_m \to \psi \in C(\Delta, \Delta)$, with each ψ_m retractive. We must show that $\psi^2 = \psi$. We note that

$$d(\psi^2, \psi) \leq d(\psi^2, \psi \circ \psi_m) + d(\psi \circ \psi_m, \psi_m^2) + d(\psi_m^2, \psi_m) + d(\psi_m, \psi)$$

Now any function in $C(\Delta, \Delta)$ is uniformly continuous, so given $\varepsilon > 0$, choose N such that $m > N \implies d(\psi_m, \psi) < \min(\varepsilon, \delta)$,, where $d(x, y) < \delta \implies d(\psi(x), \psi(y)) < \varepsilon$. Then we must have that

$$d(\psi^2, \psi) < \varepsilon + \varepsilon + 0 + \varepsilon = 3\varepsilon$$

Hence $\psi^2 = \psi$.

Next, we show that the partial function $(K,s) \mapsto \min(K,s)$, the lexicographically minimal element in K that begins with s, is continuous for each s, and its graph is closed:

$$x = \min(K,s) \iff x \in N_s \bigwedge N_s \cap K \neq \emptyset \bigwedge \forall t, K \bigcap N_{s^\frown t} \neq \emptyset \implies x|_{|s|+|t|} \leq s^\frown t$$

Now we define for each T, our retraction ψ_T. We do this by constructing a map γ from the tree over Δ to that over T. Identify T with its tree for the sake of argument. For $x \in \Delta$, if $x \in T$, (i.e., if $T \cap N_{X|_n} \neq \emptyset$ for all $n \in \mathbb{N}$) let $\psi(x) = x$. Else, let s_x be the branch of maximal length contained in x such that $s_x \in T$. Then we map x to $\min(T, s_x)$ This is easily continuous, since if $x|_n = y|_n$ for n large enough, then $d(\min(T,s_x), \min(T,s_y))$ is also small since these share the first n elements in common.

Finally, we have

$$\psi = \psi_T \iff \psi \in R \bigwedge \forall x \forall n \big[(x|_n \in T \bigwedge x|_{n+1} \notin T) \implies \psi(x) = \min(T, x|_n)\big]$$

This function is well defined, since it maps T to retracts of T in a way that each $x \notin T$ is determined. The condition $(x|_n \in T \bigwedge x|_{n+1} \notin T)$ is clopen in $\Delta \times \mathcal{K}(\Delta)$, and equality to some value for a fixed x is closed in $C(\Delta, \Delta)$. Hence the quantified condition is closed, so the graph is closed.

$\square$

The above two lemmas enable a Borel map that maps K to the retract ψ_{T_K}. We want, however, to find a way to map K to surjective continuous maps that induce isometric embeddings. To do this, we show the following:

Lemma 4.1.6. *Let K and L be compact, and let X be Polish. Then the composition function $\circ : C(K,L) \times C(L,X) \to C(K,X)$ where $(g,f) \mapsto f \circ g$, is continuous. Consequently, the map $\psi \mapsto \phi \circ \psi$. where ψ is a retraction and ϕ the surjective continuous map defined above, is continuous.*

Proof. Let $f_m \to f$ and $g_m \to g$. Let $\varepsilon > 0$. Suppose for all $m \geq N$, $d(f_m, f) < \min(\delta, \varepsilon)$ and $d(g_m, g) < \min(\delta, \varepsilon)$, where δ is such that for $x, y \in L$, if $d(x,y) < \delta$, then $d(f(x), f(y)) < \varepsilon$ (this is possible since f is a continuous map with a compact domain). Then $d(f_m \circ g_m, f \circ g) \leq d(f_m \circ g_m, f, \circ g_m) + d(f \circ g_m, f \circ g) < 2\varepsilon$.

□

Now, recalling that $\phi \in C(\Delta, \mathbb{H})$, the map $K \mapsto T_K : \mathcal{K}(\mathbb{H}) \to \mathcal{K}(\Delta)$, and $\psi : \mathcal{K}(\Delta) \to C(\Delta, \Delta)$ are all at least Borel, we define the map $\Phi : \mathcal{K}(\mathbb{H}) \to C(\Delta, \mathbb{H})$, with $\Phi_K := \phi \circ \psi_{T_K}$. By Lemma 4.1.6, the map $K \mapsto \Phi_K$ is Borel. Furthermore, the range of Φ_K is K itself, since $\Phi_K[\Delta] = \phi[\psi_{T_K}[\Delta]] = \phi[T_K] = K$, by definition of ϕ, so for any closed $K \subseteq K' \subseteq \mathbb{H}$ and for any $g \in C(K')$, the composition $g \circ \Phi_K \in C(\Delta)$ is well-defined. So Φ_K induces a lattice isometry from a $C(K)$ into $C(\Delta)$ with $g \mapsto g \circ \Phi_K$ for all $g \in C(K)$. Now we are ready to prove the theorem.

Theorem 4.1.7. *The map $\mathfrak{C} : \mathcal{K}(\mathbb{H}) \to BL(C(\Delta))$, where*

$$\mathfrak{C}(K) = \{f \in C(\Delta) : f = g \circ \Phi_K, g \in C(K)\},$$

is Borel. Consequently, we have a map from K to an isometric copy of $C(K)$.

Proof. We can suppose that Δ and any compact metric K are closed subsets of $\mathbb{H}$. We show the graph is Borel by proving that

$$E = \mathfrak{C}(K) \iff \forall m (\exists f \in C(\mathbb{H})(\psi_m(E) = f \circ \Phi_K) \bigwedge \tag{4.1}$$

$$\forall m \ \psi_m(C(\mathbb{H})) \circ \Phi_K \in E. \tag{4.2}$$

Assuming the relation in fact holds, by Lemma 4.1.6, composition of functions as displayed in the relation is Borel, so the graph is analytic, hence by [46, Theorem 14.12], the function is Borel. Now we need to show that the equivalence in lines 4.1-4.2 holds. By the discussion before the statement of the theorem, $\mathfrak{C}(K)$ is a lattice isometric copy of $C(K)$.

First, if $E = \mathfrak{C}(K)$, then clearly for all $m \in \mathbb{N}$, since the image of Φ_K is K, for all $m \in \mathbb{N}$, $\psi_m(C(\mathbb{H})) \circ \Phi_K = \psi_m(C(\mathbb{H}))|_K \circ \Phi_K \in E$. in addition, any $\psi_m(E) = g \circ \Phi_K$ with $g \in C(K)$. As $K \subseteq \mathbb{H}$, use Tietze's extension theorem to extend g to a function $g' \in C(\mathbb{H})$ where $g'|_K = g$. Thus E satisfies the relation starting line in 4.1.

Now suppose E satisfies the same relation. Let $g \in C(K)$, and consider $g \circ \Phi_K \in \mathfrak{C}(K) \subseteq C(\Delta)$. Again, by Tietze's extension theorem, let $g' \in C(\mathbb{H})$ be such that

$g'|_K = g$. Now the above relation implies that there is a sequence $g_m \to g'$ of $C(\mathbb{H})$, where we have $g_m \circ \Phi_K \in E$. Then for all $\varepsilon > 0$, we have some N such that $m > N \implies d(g_m, g) < \varepsilon$, which implies that $d(g_m \circ \Phi_K, g' \circ \Phi_K) < \varepsilon$. Note though that the image of Φ_K is K, so $g_m \circ \Phi_K \to g' \circ \Phi_K = g \circ \Phi_K \in E$. Hence $\mathfrak{C}(K) \subseteq E$. Now suppose $g \in E$, and suppose $g_m \to g$ with $g_m = \psi_{n_m}(E)$. Find $f_m \in C(\mathbb{H})$ with $g_m = f_m \circ \Phi_K$. We then want f such that $g = f \circ \Phi_K$. Observe that if $(g_m)_m$ is Cauchy, then so is $(f_m|_K)_m \subseteq C(K)$: indeed, we have

$$\begin{aligned}
d(g_m, g_n) &= d(f_m \circ \Phi_K, f_n \circ \Phi_K) \\
&= \sup\{|f_m \circ \Phi_K(x) - f_n \circ \Phi_K(x)| : x \in \Delta\} \\
&= \sup\{|f_m(y) - f_n(y)| : y \in K\} \\
&= d(f_m|_K, f_n|_K),
\end{aligned}$$

since Φ_K surjects onto K. Hence $f_m|_K \to f \in C(K)$, so $E \subseteq \mathfrak{C}(K)$. Thus we have $E = \mathfrak{C}(K)$.

□

Theorem 4.1.8. *The isometry relation on Banach lattices $I_{\cong}$ is Borel bireducible to the universal relation for orbit equivalence relations induced by Polish groups on standard Borel spaces.*

Proof. By Theorem 4.1.3, $I_{\cong}$ is reducible to an orbit equivalence relation induced by a Polish group. By [72, Theorem 1] and Theorem 4.1.7, it also reduces every such group. □

4.2 The lattice isomorphism equivalence relation

In this short section, we show that the isomorphism equivalence relation $I_{\sim}$ on separable Banach lattices is analytically complete, meaning that any analytic equivalence relation is Borel reducible to it. The technique used here is based on a portion of those found in [57], of which a central result was that the linear isomorphism relation on Banach spaces is analytically complete.

Definition 4.2.1. *Let (R_1, R_2) and (S_1, S_2) be two pairs of binary relations on standard Borel spaces X and Y respectively. A Borel map $f : X \to Y$ is a* ***Borel homomorphism from*** *(R_1, R_2)* ***to*** *(S_1, S_2) if for all $x, y \in X$, $xR_1y \to f(x)S_1f(y)$, and*

$xR_2y \to f(x)S_2f(y)$. We say that (R_1, R_2) is ***Borel hom-reducible*** *to (S_1, S_2), and write $(R_1, R_2) \preccurlyeq_B (S_1, S_2)$.*

Note that given equivalence relations R and S,

$$R \leq_B S \iff (R, \neg R) \preccurlyeq_B (S, \neg S)$$

We can also use the notation, $(R_1, R_2) \leq_B R$ and $R \leq (R_1, R_2)$ to mean $(R_1, R_2) \preccurlyeq_B (R, \neg R)$ and $(R, \neg R) \leq (R_1, R_2)$. Given the definition above, we say that, given a class $\mathcal{C}$ of binary relations, a pair (R_1, R_2) is $\mathcal{C}$-hard if any element in $R \in \mathcal{C}$ is hom-reducible to the pair (R_1, R_2). The pair is complete if $R_1, R_2 \in \mathcal{C}$. Note that if an analytic equivalence relation is hard for analytic quasi-orders, then it should be analytically complete as equivalence relations are all quasi-orders (in fact, the converse is also true: see [57,]).

We show that the isomorphism relation on Banach lattices is analytically complete by showing that it hom-reduces a pair of quasi-orders that is analytically complete. First, we construct said quasi-orders. The construction is that found in section 5 of[57], but it will be reproduced here. Consider an encoding $\alpha \to A_\alpha$ from some Polish X to the Σ^1_1 subsets of Δ. We now define binary relations $\equiv_{\Sigma^1_1}$, $\subseteq_{\Sigma^1_1}$, and pair of elements in $\mathcal{C}$.

$$\begin{aligned}
a \equiv_{\Sigma^1_1} \beta &\iff A_\alpha \neq \emptyset \bigwedge A_\beta \neq \emptyset \bigwedge A_\alpha = A_\beta \\
\alpha \subseteq_{\Sigma^1_1} \beta &\iff A_\alpha \neq \emptyset \bigwedge A_\beta \neq \emptyset \bigwedge A_\alpha \subseteq A_\beta \\
\alpha \ \mathrm{Dist}_{\Sigma^1_1} \beta &\iff A_\alpha \neq \emptyset \bigwedge A_\beta \neq \emptyset \bigwedge A_\alpha \cap A_\beta = \emptyset
\end{aligned}$$

Let $s, t \in \mathcal{S} = \mathbb{N}^{<\mathbb{N}}$, and define $s \leq t \iff |s| = |t|$ and for all $0 \leq i \leq |s|$, we have $s(i) \leq t(i)$. Now let $\mathcal{T}$ be the class of **normal trees** on $\overline{2} \times \mathbb{N}$. As a convenience, if $((u_1, s_1), ..., (u_n, s_n)) \in (\overline{2} \times \mathbb{N})^{<\infty}$, we simply denote this branch as (u, s) with $u = (u_1, ..., u_n)$ and $s = (s_1, ..., s_n)$. Similarly, if $((u_1, s_1), (u_2, s_2), ...)$ is a sequence of tuples over $\overline{2} \times \mathbb{N}$, we will equate the sequence with $(u, s) \in \Delta \times \mathcal{N}$, with $u = (u_n)_n$ and $s = (s_n)_n$. A normal tree is a non-empty tree T such that for $(u, s) \in T$ and $s \leq t$, then $(u, t) \in T$. We can look at the class of pruned normal subtrees as a subset of $2^{(2\times\omega)^{<\omega}}$. This set is closed, and hence Polish. For any normal tree T, we let

$$A(T) = \{\alpha \in \Delta : \exists\beta \in \mathcal{N} \ \big((\alpha, \beta) \in [T]\big) \ \},$$

Where $[T] \subseteq \Delta \times \mathcal{N}$ is the set of infinite branches in T. We now include the following well known result:

Lemma 4.2.2 (folklore)**.** *Any analytic subset $A \subseteq \Delta$ is equal to $A(T)$ for some normal tree T on $\overline{2} \times \mathbb{N}$.*

Proof. Since closed subsets of $\Delta \times \mathcal{N}$ correspond to pruned trees in $(\overline{2} \times \mathbb{N})^{<\infty}$, every Σ^1_1 subset of Δ is some $A(T)$ for some T (see, for instance, [46, Exercise 14.3]), so we have an encoding of the analytic sets of Δ. Let $NT = \{(s, t + t') : (s, t) \in T \text{ and } t' \in \mathbb{N}^{|t|}$, where $t + t' = (t_1 + t'_1, ..., t_n + t'_n)$. Clearly NT is normal. We now show that $A = A(T) = A(NT)$. Indeed, $A(T) \subseteq A(NT)$. Suppose now that $\alpha \in A(NT)$. Pick $\beta \in \mathcal{N}$ such that $(\alpha, \beta) \in [NT]$. Now for each n, there are $t, t' \in \mathbb{N}^n$ such that $(\alpha|_n, t) \in T$ and $t + t' = \beta|_n$. Now the tree $\{ \ (\alpha|_n, t) : t \leq \beta|_n, n \in \mathbb{N}\}$ is finitely branching, and it contains infinitely many branches from T. Thus there is an infinite branch in $[T]$ of the form (α, β'), so $\alpha \in A(T)$. □

Thus we have an encoding of the analytic sets of Δ over normal trees in $\overline{2} \times \mathbb{N}$. We also include the following definitions of relations:

Definition 4.2.3. *For $S, T \in \mathcal{T}$, we let:*

$$\begin{aligned}
S \leq_{\Sigma^1_1} T &\iff \exists\alpha \in \mathcal{N} \ \forall(u, s)((u, s) \in S \to (u, s + \alpha_{|s|}) \in T) \\
S \equiv_{\Sigma^1_1} T &\iff S \leq_{\Sigma^1_1} T \bigwedge T \leq_{\Sigma^1_1} S \\
S \neq_{\Sigma^1_1} T &\iff A(S) \neq A(T)
\end{aligned}$$

The relations $\equiv_{\Sigma^1_1}$ and $\neq_{\Sigma^1_1}$ are of interest here, since by [35, Theorem 10], the pair $(\equiv_{\Sigma^1_1}, \neq_{\Sigma^1_1})$ is complete for analytic equivalence relations. Let $\mathbf{T}$ be the complete tree on $(2 \times \omega)^{<\omega}$. To prove complete analycity of the lattice isomorphism relation, we employ the space $\mathbf{T}_2$ and notation used to define it as found in [35, Section 6].

Start with a **Cantor Scheme** $(I_u)_{u \in 2^{<\omega}}$ of closed, mutually disjoint sub-intervals of $(1, 2)$ such that the following conditions hold:

1. $I_{u^\frown 0} \cup I_{u^\frown 1} \subseteq I_u$

2. $\max I_{u^\frown 0} < \min I_{u^\frown 1}$

3. $\min I_u = \min I_{u\frown 0}$ and $\max I_u = \min I_{u\frown 1}$,

4. For all $u \neq \emptyset$, the standard unit bases for $\ell^{|u|}_{\min I_u}$ and $\ell^{|u|}_{\max I_u}$ are 2-equivalent.

We can then use $(I_u)_{u\in 2^{<\omega}}$ to homeomorphically embed a cantor set into $(1,2)$, with $\alpha \in \Delta$ mapped to the unique point p_α in $\bigcap_{u\subseteq\alpha} I_u$. Observe also that lexicographical ordering is preserved in this embedding. Consider now the vector space $c_{00}(\mathbf{T})$, with basis $(e_t)_{t\in\mathbf{T}}$. Let $\mathbf{s} \subseteq \mathbf{T}$ denote finite segments of $\mathbf{T}$, that is sets of the form $\mathbf{s} = \{t \in \mathbf{T} : t_0 \subseteq t \subseteq t_1\}$, and for a finitely supported vector $x = \sum_{t\in\mathbf{T}} \lambda_t e_t$ and $\mathbf{s} = ((u_0, s_0), ..., (u_n, s_m))$, let

$$\|\sum_{t\in\mathbf{T}} \lambda_t e_t\|_{\mathbf{s}} = \sup_{m\leq n} \|\sum_{i=0}^{m} \lambda_{(u_i,s_i)} v\delta_i\|_{\min I_{u_m}},$$

where $(\delta_i)_i$ is the standard basis for $\ell_{\min I_{u_m}}$. Each $\|\cdot\|_{\mathbf{s}}$ is a lattice semi-norm on $c_{00}(\mathbf{T})$. Furthermore, note that for $m \leq n$, we have $I_{u_m} \subseteq I_{u_n}$, and $\ell^n_{\min I_{u_n}}$ is 2-equivalent to $\ell^n_{\min I_{u_m}}$. Thus

$$\|\sum_{i=0}^{m} \lambda_{(u_i,s_i)}\delta_i\|_{\min I_{u_m}} \leq 2\|\sum_{i=0}^{n} \lambda_{(u_i,s_i)}\delta_i\|_{\min I_{u_n}}$$

More generally, if $(\alpha, \beta) \in [\mathbf{T}]$ contains $\mathbf{s}$ as a sub-segment, then $p_\alpha \in I_{u_n}$ and $\ell^n_{p_\alpha}$ is 2-equivalent in norm to $\ell_{\min I^n_{u_n}}$. Now let

$$\left\|\sum_{t\in\mathbf{T}} \lambda_t e_t\right\| = \sup\left\{ \left(\sum_{i=1}^{n} \left\|\sum_{t\in\mathbf{s}_i} \lambda_t e_t\right\|^2_{\mathbf{s}_i}\right)^2 : (\mathbf{s}_i)^n_{i=1} \text{ is an admissable set of segments}\right\},$$

And let $\mathbf{T}_2$ be the closure of $c_{00}(\mathbf{T})$ under $\|\cdot\|$. Note that the norm on $\mathbf{T}_2$ is a lattice norm, so $\mathbf{T}_2$ is an (atomic order continuous) lattice. Before we prove the main result, we state without proof [35, Lemma 14]:

Lemma 4.2.4. *Let S, T be subtrees of $\mathbf{T}$ and let $\phi : S \to T$ be an isomorphism of trees such that for all $(u, s) \in \mathbf{T}$, $\phi(u, s) = (u, s')$ with $(u, s') \in T$. Let also Z_T and Z_S be the sublattices of $\mathbf{T}_2$ generated by the atoms e_t with $t \in T$ and S respectively. Then the map*

$$M_\phi : e_{(u,s)} \mapsto e_{\phi(u,s)}$$

induces a lattice isometry between Z_T and Z_S.

Let $\mathcal{I}(\mathcal{V})$ be the set of infinite dimensional ideals in the Pelczynski lattice $\mathcal{V}$ with generating atoms $(e_n)_n$. By 3.3.1, this set is a standard Borel space. Observe also that each $X \in \mathcal{I}$ is generated by a subsequence $(e_{n_i})_i)$, and that this sequence is an unconditional basic sequence. Based on this, we repeat an observation made prior to [35, Theorem 15] whose source is [62]: that if $(x_i)_i$ and $(y_i)_i$ are unconditional basic sequences and there are injective $f, g : \mathbb{N} \to \mathbb{N}$ such that $(x_i)_i$ is equivalent to $(y_{f(i)})$ and $(y_i)_i$ is equivalent to $(x_{g(i)})_i$, then $(x_i)_i$ is equivalent to $(y_{\sigma(i)})_i$ for some $\sigma \in S_\infty$. Translated to the lattice setting, what this means is that for $X, Y \in \mathcal{I}(\mathcal{V})$, if X ideally embeds into Y, and Y into X, then X is lattice isomorphic to Y. Let $\sim_{\mathcal{V}}$ be the lattice isomorphism relation on the lattices in $\mathcal{I}(\mathcal{V})$.

We now show the following:

Theorem 4.2.5. *The lattice isomorphism relation for separable Banach lattices is a complete analytic equivalence relation.*

Proof. We show that $\sim_{\mathcal{V}}$ reduces $(\equiv_{\Sigma_1^1}, \neq_{\Sigma_1^1})$ on pruned normal trees over $2 \times \omega$. By [35, Theorem 11], the restriction of $(\equiv_{\Sigma_1^1}, \neq_{\Sigma_1^1})$ to pruned normal trees is still analytically complete for equivalence relations.

For a pruned normal tree S, let $\phi(S) = BL(\{e_t : t \in S\})$, where $(e_t)_{t \in \mathbf{T}}$ is a subsequence of atoms of $\mathcal{V}$ generating an isomorphic copy of $\mathbf{T}_2$. Suppose now that $S \equiv_{\Sigma_1^1} T$, and let $\alpha \in \mathcal{N}$ be such that for all $(u, s) \in S$, $(u, s + \alpha|_{|s|}) \in T$. Then by Lemma 4.2.4 the map $e_{u,s} \mapsto e_{u,s+\alpha|_{|s|}}$ induces a lattice isometric embedding from $\phi(S)$ into $\phi(T)$ mapping atoms to atoms. Similarly, there is some $\beta \in \mathbb{N}$ such that for all $(u, s) \in T$, $(u, s + \beta|_{|s|}) \in S$, so $\phi(T)$ also ideally embeds into $\phi(S)$. Then $\phi(S)$ is lattice isomorphic to $\phi(T)$.

Suppose now that $S \neq_{\Sigma_1^1} T$, and pick some $\alpha \in \Delta$ such that $\alpha \in A(S) \setminus A(T)$. Then ℓ_{p_α} ideally embeds into $\phi(S)$, but it cannot embed ideally isomorphically into $\phi(T)$. Indeed, if $(e_s)_{s \in A} \subseteq (e_t)_{t \in T}$ is a subsequence, then there is an infinite subset $B \subseteq A$ such that $(e_s)_{s \in B}$ is either a subsequence of $(e_{\gamma|_n, \beta|_n})_n$ for some $(\gamma, \beta) \in [T]$ or is an anti-chain. If the first is true, then $(e_s)_{s \in B}$ induces a lattice isomorphic to ℓ_{p_γ} with $p_\gamma \neq p_\alpha$, since otherwise $\alpha \in A(T)$. If B is an anti-chain, then any branch in $\mathbf{T}$ goes through at most one element in B, so the lattice induced by $(e_s)_{s \in B}$ is lattice isomorphic to ℓ_2. Either way, the lattice generated by $(e_s)_{s \in A}$ is not isomorphic to ℓ_{p_α}, so $\phi(S)$ is not lattice isomorphic to $\phi(T)$. $\square$

4.3 Questions and further research

Question 4.3.1 (r.i. Banach lattices)**.** We know that the isomorphism and isometry equivalence relations restricted to rearrangement invariant lattices is Borel, making it strictly simpler than the general equivalence relations of lattice isomorphism and lattice isometry. However, we do not know where they lie in the Borel Hierarchy.

Question 4.3.2 (Lattices with very few sublattices)**.** One can also restrict the isomorphism and isometry relations to sublattices of some given lattice X. In some instances, such as $BL(\ell_p)$, there exists only one infinite dimensional sublattice up to isometry or isomorphism, which is ℓ_p, while the lattices in $BL(L_p)$ are isomorphically and isometrically determined by the numbers of atoms. What sublattices contain very few sublattices up to isometry and isomorphism?

Since ℓ_p and L_p are both very "simple" lattices containing only one or at most, countably many lattices, one can also increase the difficulty by discerning the complexity of the isometry and isomorphism equivalence relations for "next simplest" lattices such as $\ell_p \oplus \ell_q$. Some preliminary but unfinished exploration shows that these lattices have continuum many non-isomorphic sublattices.

Question 4.3.3 (r.i. lattices and reducibility)**.** This is more of a Descriptive Set theory rather than a Banach lattice question, but it may have some ties here. For $1 \leq p \leq \infty$, we can define the following equivalence relation $\sim_p$ on $\mathbb{R}^{\mathbb{N}}$, with $(x_n) \sim_p (y_n) \iff (x_n - y_n) \in \ell_p$. In the case where $p = 1$, this relation is equivalent to the relation E_2 on Δ with $xE_2y \iff \sum_{x_k \neq y_k} \frac{1}{k} < \infty$.

The main result in [27] holds that

$$\sim_p \leq_B \sim_q \iff p \leq q.$$

Observe that when $p < q$, then $\sim_p$ is strictly reducible to $\sim_q$. We can generalize the question by considering the equivalence relation E_X on $\mathbb{R}^{\mathbb{N}}$ induced by some atomic lattice X, viewed as a subset of $\mathbb{R}^{\mathbb{N}}$, where given $x, y \in \mathbb{R}^{\mathbb{N}}$, we have

$$xE_Xy \iff x - y \in X$$

Now the general question might be too complicated, but we can limit the question specifically to r.i. lattices or even Orlicz sequence spaces. Some research has already been done on this (see [26]), but there is still much room for exploration. For example, given two Orlicz sequence spaces X and Y, under what conditions is $E_X \leq_B E_Y$?

Chapter 5

Banach lattices, universality, and homogeneity

5.1 Introduction

The work for this chapter is largely drawn from [70]. Here, we give a constructive proof of the Amalgamation Property in Banach lattices and construct a "Gurarij Lattice" $\mathfrak{BL}$ using various techniques.

Some comparable results are already known in Banach space theory. The Gurarij space $\mathfrak{G}$ is an isometrically universal separable Banach space with the following homogeneity property: for any finite dimensional spaces $A \subseteq B$, any isometric embedding $f : A \to \mathfrak{G}$, and any $C > 1$, there exists a map $g : B \to \mathfrak{G}$ extending f such that $\frac{1}{C}\|x\| \leq \|g(x)\| \leq C\|x\|$ for all $x \in B$. Such separable Banach spaces are isometrically unique (see [59], as well as a simplified proof by Kubiś and Solecki in [49]). An alternate construction by Ben Yaacov characterizing $\mathfrak{G}$ as a metric Fraïssé limit is found in [13]. As a Fraïssé limit, $\mathfrak{G}$ has another kind of homogeneity that strengthens isomorphic embeddings with small distortion to isometric embeddings while sacrificing full commutativity. In other words, given finite dimensional spaces $A \subseteq B$, an isometric embedding $f : A \to \mathfrak{G}$, and $\varepsilon > 0$, there exists an isometric embedding $g : B \to \mathfrak{G}$ such that $\|f - g|_A\| < \varepsilon$. Since $\mathfrak{G}$ is a Fraïssé limit, it is also isometrically unique among separable spaces with this property.

We prove the lattice analogue of the above stated result. Using Fraïssé machinery, we show that there is a unique isometrically universal separable Banach lattice $\mathfrak{BL}$ with the following homogeneity property: for any lattices $A \subseteq B$ generated by finitely many elements and lattice isometric embeddings $f : A \to \mathfrak{BL}$, for all $(a_1, ..., a_n) \subseteq A$ generating A, and for all $\varepsilon > 0$, there exists a lattice isometric embedding $g : B \to \mathfrak{BL}$ such that for each a_i, $\|f(a_i) - g(a_i)\| < \varepsilon$ (Theorem 5.4.1). The key to this result is the fact that Banach lattices have the Amalgamation Property (Theorems 5.3.8 and 5.3.10). Observe that if A and B are finite dimensional, we can strengthen almost commutativity of the diagram restricted to generators to almost commutativity in norm. $\mathfrak{BL}$ can also be constructed as an inductive limit of $\ell_\infty^m(\ell_1^n)$ lattices, paralleling the construction of the Gurarij space as a limit of ℓ_∞^n spaces (Theorem 5.4.3).

$\mathfrak{BL}$ does not have the homogeneity property that originally characterized $\mathfrak{G}$, however, because in certain cases one cannot extend a lattice isometric embedding in a way that preserves both lattice structure and full commutativity in the separable setting. In addition, even though $\mathfrak{G}$ can be "almost" homogeneous in either of the forms mentioned above, it cannot fully homogeneous in the sense of requiring both isometric embeddings and full commutativity of the diagram. Since it is unique, no separable spaces can have this stronger property. There exist non-separable Banach spaces, however, that are fully homogeneous, not just for the class of finite dimensional spaces, but also for separable spaces. Such spaces, referred to as spaces of universal disposition, are constructed by Avilés, Sánchez, Castillo, and Moreno in [7]. A different construction (which assumes the CH) using Fraïssé sequences is given in [47], where uniqueness is also established. Very recently, Avilés and Tradecete also constructed a (necessarily non-separable) lattice of universal disposition for separable lattices [8].

Homogeneity in sublcasses of Banach lattices has been recently explored at length by Ferenczi, Lopez-Abad, Mbombo, and Todorcevic [34]. This paper treats on various levels of homogeneity in L_p Banach spaces, but it also explores lattice homogeneity. Specifically, for $1 \leq p < \infty$, the separable spaces $L_p(0,1)$ are Fraïssé limits for ℓ_p^n spaces with lattice embeddings as corresponding maps. The authors also construct an approximately ultra-homogeneous M-space for the class of finite dimensional M spaces.

Outside of the Banach lattice setting, homogeneous structures have been found for various classes. Using injective objects, Lupini proved the existence of homogeneous structures for the classes of function systems, p-multinormed spaces, and M_q-spaces [58]. Certain C^*-algebras can also be constructed as Fraïssé limits of appropriate classes with relaxed conditions, including all UHF algebras, the hyperfinite II_1-factor [28], the Jiang-Su algebra [60], and more recently, a projectively universal AF-algebra constructed in [40].

5.2 Preliminaries

We start with some concepts largely taken from [13]: let $\mathcal{L}$ be a collection of symbols. These can be either **predicate symbols** or **function symbols**. Each predicate or function symbol has an associated number called its **arity**. We then call $\mathfrak{A}$ with associated metric space A an **$\mathcal{L}$-structure** if

1. For every predicate symbol R with arity n, there is a continuous interpretation $R^{\mathfrak{A}} : A^n \to \mathbb{R}$. We can also consider the distance to be a binary symbol (found in every structure).

2. For every function symbol f with arity n, we have a continuous interpretation $f^{\mathfrak{A}} : A^n \to A$. Note that if a function symbol c has 0-arity, then it is a **constant symbol**, and $c^{\mathfrak{A}} \in A$.

These are different from the typical definitions of $\mathcal{L}$-structures in continuous logic as found in [14], the latter which require uniform continuity for functions and predicates but do not require that X and Y be bounded. The theory of Banach lattices can be formulated in the language $\mathcal{L} = (+, \mathbb{R}, \wedge, \vee)$. In particular, its function and predicate symbols have corresponding moduli of uniform continuity which are independent of their interpretation in a particular lattice. Given $\overline{x} = (x_1, ..., x_n)$, $\overline{y} = (y_1, ..., y_n) \subseteq X$, where X is a metric space, we let

$$d(\overline{x}, \overline{y}) = \max_{i \leq n} d(x_i, y_i).$$

As in [14, Chapter 2], we define the modulus of uniform continuity. A function $\Delta_f : \mathbb{R}^+ \to (0, 1]$ is a **modulus of uniform continuity** for a $\mathcal{L}$-function or predicate symbol f of arity n if for all $\mathcal{L}$-structures $\mathcal{M}$ and $\overline{x}, \overline{y} \in \mathcal{M}^n$, $d(\overline{x}, \overline{y}) < \Delta_f(\varepsilon)$ implies that $d(f^{\mathcal{M}}(\overline{x}), f^{\mathcal{M}}(\overline{y})) < \varepsilon$. For example, the function symbol $\wedge$ has modulus of continuity $\Delta(\varepsilon) = \frac{1}{4}\varepsilon$. That is, given a lattice X and (x_1, y_1) and $(x_2, y_2) \in X^2$, if $d((x_1, y_1), (x_2, y_2)) < \frac{1}{4}\varepsilon$, then $\|x_1 \wedge y_1 - x_2 \wedge y_2\| < \varepsilon$. The definition of moduli of continuity in the appendix in [14, Chapter 2] assumes that moduli have domains restricted to $(0, 1]$ but we can extend such functions to $\mathbb{R}^+$ by letting $\Delta_f(r) = \Delta_f(1)$ for all $r > 1$. Propositions 2.4 and 2.5 in [14] show that compositions of uniformly continuous real functions and $\mathcal{L}$-function and $\mathcal{L}$-predicate symbols also have corresponding moduli of uniform continuity, since they are also uniformly continuous.

We say that A is a **substructure** of B if A is a closed subset of B which is also closed under all combinations of the function symbol operations. For Banach lattices, X is a substructure of Y if it is a sublattice of Y. Let $f : A \to B$ be a map between two $\mathcal{L}$ structures. If f preserves norms, function operations, and predicate symbols in $\mathcal{L}$, then f is considered an **embedding**. Observe that if the structures in question are Banach lattices, then embeddings in the sense of logic structures is none other than lattice embeddings as defined in Section 1.2. Throughout, we will also be making use of C-embeddings and C-isometries, and unless otherwise noted, by (C)-embeddings and (C)-isometries we mean lattice isomorphic embeddings and isometries, and not merely linear maps.

Let $A_0 \subseteq B$. We then let $< A_0 >$ be the substructure generated by A_0. This can be understood as the smallest set $A \subseteq B$ with $A_0 \subseteq A$ and A a substructure of B. We say that A is **finitely generated** if there exist $(a_1, ...a_n) \subseteq A$ such that $A =< (a_1, ..., a_n) >$. Suppose that $\mathcal{K}$ is a class of finitely generated $\mathcal{L}$-structures. If $A \in \mathcal{K}$, we say that A is a $\mathcal{K}$**- structure** if every finitely generated substructure of A is also in $\mathcal{K}$.

For $A =< \overline{a} >$ and $B =< \overline{b} >$ with $|\overline{a}| = |\overline{b}|$, we define

$$d^{\mathcal{K}}(\overline{a}, \overline{b}) = \inf_{\substack{\phi_1 : A \to C \\ \phi_2 : B \to C}} d(\phi_1(\overline{a}), \phi_2(\overline{b})),$$

where ϕ_1 and ϕ_2 are both embeddings into some ambient $\mathcal{K}$-structure C. If we clearly understand generating tuples $\overline{a}$ and $\overline{b}$ for lattices A and B to be in some larger ambient space without necessary reference to explicit embeddings, we just write $d(\overline{a}, \overline{b})$ instead of $d(\phi_1(\overline{a}), \phi_2(\overline{b}))$.

Let $\mathcal{K}$ be a class of finitely generated structures. We then say $\mathcal{K}$ is **Fraïssé** if:

- $\mathcal{K}$ has the *Hereditary Property* (HP): every member of $\mathcal{K}$ is a $\mathcal{K}$-structure.
- $\mathcal{K}$ has the *Joint Embedding Property* (JEP): any two $\mathcal{K}$-structures embed into a third. (Note that if $\mathcal{K}$ has the JEP, then $d^{\mathcal{K}}$ is defined for all pairs of tuples in $\mathcal{K}$ of the same length.)
- $\mathcal{K}$ has the *Near Amalgamation Property* (NAP): for any structures $A =< \overline{a} >$, B_1 and B_2 in $\mathcal{K}$ with embeddings $f_i : A \to B_i$, and for all $\varepsilon > 0$, there exists a

$C \in \mathcal{K}$ and embeddings $g_i : B_i \to C$ such that

$$d(g_1 \circ f_1(\overline{a}), g_2 \circ f_2(\overline{a})) < \varepsilon.$$

If $g_1 \circ f_1 = g_2 \circ f_2$, then we just say that $\mathcal{K}$ has the Amalgamation Property (AP). Clearly the AP implies the NAP.

- $\mathcal{K}$ has the *Polish Property* (PP): if $\mathcal{K}$ has the JEP, HP and NAP, then $d^{\mathcal{K}}$ is a pseudo-metric over $\mathcal{K}$. If $d^{\mathcal{K}}$ is separable and complete in $\mathcal{K}_n$ (the $\mathcal{K}$-structures generated by n many elements):

- $\mathcal{K}$ has the *Continuity Property* (CP): every symbol in $\mathcal{L}$ is continuous on $\mathcal{K}$: that is, for function symbols, the map $(\overline{a}, \overline{b}) \mapsto (\overline{a}, \overline{b}, f^{(\overline{a})}(\overline{a}))$ is continuous, and for predicate symbols P, the map $\overline{a} \mapsto P^{\overline{a}}(\overline{a})$ is continuous.

By [13, Theorem 3.21], if $\mathcal{K}$ is Fraïssé, there exists a separable space $\mathfrak{M}$, known as the **Fraïssé limit**, that is universal for $\mathcal{K}$ and **approximately ultra-homogeneous** on $\mathcal{K}$. That is, for all finitely generated structures $A =< \overline{a} > \subseteq \mathfrak{M}$, embeddings $f : A \to \mathfrak{M}$, and $\varepsilon > 0$, there exists an automorphism $\phi : \mathfrak{M} \to \mathfrak{M}$ such that $d(f(\overline{a}), \phi(\overline{a})) < \varepsilon$. Conversely, if a space $\mathfrak{M}$ is approximately ultra-homogeneous, its finitely generated substructures form a Fraïssé class, and $\mathfrak{M}$ is its limit. Such a space is also isometrically universal for all separable $\mathcal{K}$ structures (including those which are not finitely generated).

Instead of the PP and CP, a class $\mathcal{K}$ may have the following weakened conditions:

- The *Weak Polish Property* (WPP): the metric $d^{\mathcal{K}}$ is separable (but not necessarily complete)

- The *Cauchy Continuity Property* (CCP): the map $(\overline{a}, \overline{b}) \mapsto (\overline{a}, \overline{b}, f^{(\overline{a})}(\overline{a}))$ sends $d^{\mathcal{K}}$- Cauchy sequences to Cauchy sequences, and for predicate symbols P, the map $\overline{a} \mapsto P^{\overline{a}}(\overline{a})$ sends Cauchy sequences to Cauchy sequences,

If $\mathcal{K}$ has the HP, JEP, and NAP in addition to the two conditions above, then $\mathcal{K}$ is an **incomplete Fraïssé class**. A relevant example is that of finite dimensional ℓ_p spaces ([13, Section 4.2] gives a brief discussion). These have a (unique) Fraïssé limit of their completion, which is the class of separable L_p spaces, and the limit is $L_p(0, 1)$. See [34, Proposition 3.7] for a recent proof using tools from analysis.

The notion of universal disposition in Banach space theory can also be studied with Banach lattices. Let $\mathcal{C}$ be a class of Banach lattices. A lattice X is of **approximately universal disposition** for a class $\mathcal{C}$ with lattices defined by finitely many elements if for all $A \in \mathcal{C}$ and for all embeddings $f : A \to X$, $g : A \to B$, with $A \in \mathcal{C}$ defined by $\overline{a}$ and $B \in \mathcal{C}$, and for all $\varepsilon > 0$, there exists an $(1+\varepsilon)$-embedding $h : B \to X$ such that $\|h \circ g(\overline{a}) - f(\overline{a})\| < \varepsilon$. Approximate universal disposition relaxes the condition in approximate ultra-homogeneity of the existence of an embedding down to a $(1+\varepsilon)$-embedding for arbitrarily small ε.

Definition by finitely many elements in lattices can occur in more than one way. One can speak, for example, of finite generation in the context of the logic of metric structures. On the other hand, one might refer to finite dimensional lattices. For spaces of approximately universal disposition, if the class in question is finite dimensional lattices, we can let the finitely many atoms define the lattice's basis rather than generators doing so (in fact any finite dimensional lattice can be generated by two elements: see Theorem 5.4.4), and we can strengthen the requirement that $\|h \circ g(\overline{a}) - f(\overline{a})\| < \varepsilon$ to a norm requirement that $\|h \circ g - f\| < \varepsilon$.

Throughout the paper we rely on the notion of finite branchability. Let E be a Banach lattice. Let $(A_n)_n$ be a sequence of finite non-empty sets, and let $T = \cup_{k=0}^{\infty} \prod_{n=1}^{k} A_n$ be the tree generated by them. Suppose also that $(x_\sigma)_{\sigma \in T} \subseteq E_+$. We then say that (x_σ) is a **finitely branching tree** in E_+ if for all σ with $|\sigma| = k$, $(x_{(\sigma \frown b)})_{|\sigma|=k}$ is disjoint, and

$$x_\sigma = \sum_{b \in A_{k+1}} x_{(\sigma \frown b)}.$$

Note that given the property outlined in the definition, $\mathrm{span}(\{(x_\sigma)_{\sigma \in T}\})$ is a vector lattice in E. If $\mathrm{span}(\{(x_\sigma)_{\sigma \in T}\})$ is dense in E for some finitely branching tree (x_σ), we call E **finitely branchable.** Finitely branchable lattices allow us to reduce problems involving finitely generated, but infinite dimensional lattices to that of finite dimensional lattices, since they are inductive limits of finite dimensional lattices. It is also easy to show the other direction: If a lattice is the inductive limit of finite dimensional lattices, then it is finitely branchable. Finally, observe that finitely branchable lattices are separable.

Throughout, we will be working with two named classes of Banach lattices: let $\mathcal{K}$ be the class of finitely generated lattices, and $\mathcal{K}'$ be the class of sublattices of $\ell_\infty^m(\ell_1^n)$

spaces, with $m, n \in \mathbb{N}$. Let $\mathcal{K}_n \subseteq \mathcal{K}$ be the class of lattices generated by n elements, and likewise for $\mathcal{K}'_n \subseteq \mathcal{K}'$. Here we do not require that the generating elements be distinct or minimal. We also will make use of the isometrically universal separable lattice $\mathcal{U} := C(\Delta, L_1[0,1])$ constructed in [51]. It turns out that $\mathcal{U}$ is finitely branchable and in particular is the inductive limit of an increasing union of lattices in $\mathcal{K}'$, which will be useful later on.

We conclude this section with an outline of the rest of this paper. Section 5.3 explores the AP in Banach lattices and is split into two subsections. In the firs, We use Theorem 2.5.4 to prove an approximate amalgamation property for finite dimensional lattices (Theorem 5.3.3). In particular, it is shown that $\mathcal{K}'$ has the AP. In the second subsection, we use the results in the first subsection to show that the class of Banach lattices has the AP (Theorems 5.3.8 and 5.3.10). The key to expanding the results on $\mathcal{K}'$ is the use of finitely branchable lattices. We then end the section with some additional results on amalgamation over C-embeddings.

In Section 5.4, we prove the existence of a separable approximately ultra-homogeneous lattice $\mathfrak{BL}$ by showing that $\mathcal{K}$ is a metric Fraïssé class and explore some of its structural properties (Theorem 5.4.1). The subclass $\mathcal{K}'$ is not just the first step to amalgamation; it is itself an incomplete Fraïssé class that is dense in the class of finitely generated separable lattices according to the Fraïssé metric (Lemma 5.4.2). We use this fact to show that $\mathfrak{BL}$ is finitely branchable (Theorem 5.4.3). Finitely branchable lattices are themselves finitely generated (Theorem 5.4.4), so unlike the Gurarij space, $\mathfrak{BL}$ is finitely generated, and in particular can be generated by two elements.

In Section 5.5, we show that any separable lattice of approximately universal disposition for finitely generated lattices is isometric to $\mathfrak{BL}$ (Theorem 5.5.2). We also construct lattices of approximately universal disposition for finite dimensional lattices and show that any such lattice which is also finitely branchable is isometric to $\mathfrak{BL}$ (Theorem 5.5.4). Finally, we show a self-similarity property of $\mathfrak{BL}$: any non-trivial projection band in $\mathfrak{BL}$ is isometric to $\mathfrak{BL}$ (Theorem 5.5.5).

Finally, Section 5.6 builds on the tools in Section 5.5 to construct a Pelczynski latttice $\mathfrak{U}$ of almost universal disposition for finite dimensional lattices which are ideal in $\mathfrak{U}$. This lattice is unique up to ε-isometry for all $\varepsilon > 1$, but it is not isometrically unique. Using $\mathfrak{U}$, we can also show that Pelczynski lattices in general are not almost isometrically unique.

5.3 Banach lattices and the Amalgamation Property

The bulk of this section is dedicated to proving that the class of Banach lattices has the AP. As this paper was nearing its completion, Avilés and Tradecete independently proved that Banach lattices have the Amalgamation Property by generating pushouts using free Banach lattices (see [8, Theorem 4.4]). We give an alternative approach. We first show that $\mathcal{K}'$ itself has the AP, and then expand this result to $\mathcal{K}$ and to lattices in general.

5.3.1 The Amalgamation Property in $\mathcal{K}'$

Lattices in $\mathcal{K}'$ play a key role in subsequent results on homogeneous lattices and their structure. We present some of the notation that will be used in subsequent proofs: Suppose $F \in \mathcal{K}'$, and let $(e_1, ..., e_m)$ be the atoms of F. Let $f : F \to G := \ell_\infty^N(\ell_1^M)$ be a C-embedding, with $C \geq 1$. Let $u(k, j)$ be j'th atom in the k'th copy of ℓ_1^M. We then have, for each $e_i \in F$, that $f(e_i) = \sum_{k,j} a^i(k, j)u(k, j)$. Note that f maps atoms to disjoint positive elements, so we can just let $a^i(k, j) = a(k, j)$, and sum up only over atoms that support $f(e_i)$. Specifically, we fix a row k and let

$$F_i^k = \{j \leq M : f(e_i) \wedge u(k, j) > 0\}.$$

Then

$$f(e_i) = \sum_k \sum_{j \in F_i^k} a(k, j)u(k, j).$$

Observe that F and f induce an $N \times m$ matrix A_F^f, with $A_F^f(k, i) = \sum_{j \in F_i^k} a(k, j)$. If F_i^k is empty, then $A_F^f(k, i) = 0$. It turns out the rows of A_F^f capture F's structure completely, while small distortions in f imply small distortions in A_F^f. We give a lemma to this effect. From now on, if we have two C-isometries $f_j : F \to G_j$ with $j = 1, 2$ be C-isometries with $C \geq 1$, with G_1 and G_2 both $\ell_\infty^N(\ell_1^M)$ spaces, we just let $A = A_F^{f_1}$ and $B = A_F^{f_2}$. For $1 \leq l \leq N$, we also let $A(l) = (A(l, 1), A(l, 2), ..., A(l, m))$ and $B(l) = (B(l, 1), B(l, 2), ..., B(l, m))$.

Lemma 5.3.1. *Let $f_j : F \to G_j$ with $j = 1, 2$ be C-isometries with $C \geq 1$, and suppose G_1 and G_2 be $\ell_\infty^N(\ell_1^M)$ spaces. Then for all rows l, we have $A(l) \in C^2SCH(\{B(k) : 1 \leq k \leq N\})$. In particular, if each f_j is an embedding, then $SCH(\{B(k) : 1 \leq k \leq$*

$N\}) = SCH(\{A(k) : 1 \leq k \leq N\})$.

Proof. Let $r \in \mathbf{B}(\ell_\infty^M)_+$. Then there exists some row k such that for all rows l, $\frac{1}{C}\sum r_n B(l,n) \leq \|\sum r_n e_n\| \leq C \sum r_n A(k,n)$. Now that $B(l) \notin C^2 SCH(\{A(k) : k \leq N\})$ for some l. Then by [64, Proposition 19.7] there exists some $r \in \mathbf{B}(\ell_\infty^M)_+$ such that

$$C^2 \sup_{y \in SCH(A(k))} \sum r_n y_n < \sum r_n B(l,n) \leq C^2 \sum r_n A(k,n)$$

for some k, which is a contradiction. □

In the case that $C = 1$, recall that the construction in Theorem 2.5.4 used N rows of ℓ_1^M to correspond to the N order extreme points in the unit ball of X^*. Since we can think of the rows in A_F^f as elements in the dual space F^*, where $A(l)(\sum c_i e_i) = \sum c_i A(l,i)$, then by Lemma 5.3.1, we actually have $SCH(\{A(l)\}) = \mathbf{B}(F^*)$. In particular, any $F \in \mathcal{K}'$ has finitely many order extreme points. Combined with Lemma 2.5.3, we thus have the following result:

Corollary 5.3.2. *The four following properties are equivalent for finite dimensional lattices X:*

1. $OEP(\mathbf{B}(X))$ *is finite.*

2. $EP(\mathbf{B}(X))$ *is finite.*

3. $EP(\mathbf{B}(X^*))$ *is finite.*

4. $X \in \mathcal{K}'$.

We now prove the following:

Theorem 5.3.3. *Suppose for $j = 1, 2$, $f_j : E \to F_j$ are C-embeddings with F_1 and F_2 in $\mathcal{K}'$ with $C \geq 1$. Then there exist $G \in \mathcal{K}'$ and C-embeddings $g_j : F_j \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$. That is, the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{f_1} & F \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\ F_2 & \xrightarrow{g_2} & G \end{array}$$

In particular, $\mathcal{K}'$ has the AP.

Proof. We can assume that $F_j = \ell_\infty^N(\ell_1^{M_j})$, where $M_j = \dim F_j$ for $j = 1, 2$. Let $e_1, ..., e_n$ be the atoms in E, and for F_1 and F_2, we let $u(k,j)$ and $v(k,j)$, respectively, correspond to the j'th atom in the k'th row (that is, the k'th copy of $\ell_1^{M_j}$). For row l and atom i, we let

$$F_{1,i}^l = \{j \leq M_1 : f_1(e_i) \wedge u(l,j) > 0\},$$

and similarly, we let

$$F_{2,i}^l = \{j \leq M_2 : f_2(e_i) \wedge v(l,j) > 0\}$$

We now define g_1 and g_2. Let F_j' be the the lattice ideal in F_j generated by $f_j(E)$, and let $(F_1' \otimes F_2') \oplus F_1 \oplus F_2$ be understood as a vector lattice with atoms of the form $u(k,j) \otimes v(l,m)$, $u(k,j)$, and $v(l,m)$. For $u(k,j)$ with $j \in F_{1,k}^i$, let $g_1(u(k,j)) = u(k,j) \otimes f_2(e_i)$. If $u(k,j) \notin F_{1,k}^i$ for any i, let $g_1(u(k,j)) = u(k,j)$. For $v(l,m) \in F_{2,l}^i$, let $g_2(v(l,m)) = f_1(e_i) \otimes v(l,m)$, and if $v(l,m) \notin F_{2,l}^i$ for any i, let $g_2(v(l,m)) = v(l,m)$. First, we show that $g_1 \circ f_1 = g_2 \circ f_2$. Indeed, we have

$$\begin{aligned}
g_1 \circ f_1\Big(\sum_i c_i e_i\Big) &= \sum_i c_i g_1\Big(\sum_k \sum_{j \in F_{1,k}^i} a(k,j) u(k,j)\Big) \\
&= \sum_i c_i \Big(\sum_k \sum_{j \in F_{1,k}^i} a(k,j) u(k,j)\Big) \otimes f_2(e_i) \\
&= \sum c_i (f_1(e_i) \otimes f_2(e_i)),
\end{aligned}$$

and similarly:

$$\begin{aligned}
g_2 \circ f_2\Big(\sum_i c_i e_i\Big) &= \sum_i c_i g_2\Big(\sum_l \sum_{m \in F_{2,l}^i} b(l,m) v(l,m)\Big) \\
&= \sum_i c_i f_1(e_i) \otimes \Big(\sum_l \sum_{m \in F_{2,l}^i} b(l,m) v(l,m)\Big) \\
&= \sum c_i (f_1(e_i) \otimes f_2(e_i)).
\end{aligned}$$

Let $G = BL(g_1(F_1) \cup g_2(F_2))$, and let the unit ball of G be $SCH\big(g_1(\mathbf{B}(F_1)) \cup g_2(\mathbf{B}(F_2))\big)$.

Note that g_1 and g_2 are both contractive maps. We now show that they are also C^2-embeddings. This will be sufficient, because then we can replace g_1 and g_2 with

Cg_1 and Cg_2 while still preserving commutativity in the diagram. These latter maps are themselves C-embeddings, thus proving the theorem. Since $g_i(\mathbf{B}(F_i))$ has only finitely many order extreme points, and since the resulting space is finite dimensional, $SCH\big(g_1(\mathbf{B}(F_1)) \cup g_2(\mathbf{B}(F_2))\big)$ is also closed. So we need only to show without loss of generality that if $x, y \in \mathbf{S}(F_1)_+$, $z \in \mathbf{S}(F_2)_+$, and $rg_1(x) \leq tg_1(y) + (1-t)g_2(z)$ with $0 \leq t \leq 1$, then $r < C^2$.

Suppose x, y and z are as above. Since for some $k \leq N$, $\big\|x \wedge \big(\sum_j u(k,j)\big)\big\| = 1$ and $g_1\big(x \wedge (\sum_j u(k,j))\big) \leq g_1(x)$, we can assume that $x = \sum_j c_j u(k,j)$ with $\sum c_j = 1$. Furthermore, since

$$rg_1(x) \leq tg_1(y) \wedge rg_1(x) + (1-t)g_2(z) \wedge rg_1(x),$$

we can also assume that $tg_1(y) \leq rg_1(x)$, so $y = \sum d_j u(k,j)$, with $\sum d_j \leq 1$. Finally, we can let $z = \sum_1^M \mu_n z_n$, where z_n is an order extreme point in F_2; that is, there is a sequence $s^n = (s^n_l)_l$ of length N such that $z_n = \sum_l v(l, s^n_l)$, and furthermore, $\mu_n > 0$ with $\sum \mu_n = 1$. Then

$$0 \leq g_1(rx - ty) \leq g_2\bigg(\sum_i \sum_l \sum_{n: s^n_l = m \in F^i_{2,l}} \mu_n v(l,m)\bigg).$$

Now both sides of the inequality are supported, and the left hand side fully supported, by atoms of the form $u(k,j) \otimes v(l,m)$ where $j \in F^i_{1,k}$ and $m \in F^i_{2,l}$. Thus, for any $u(k,j) \in {F'}^{\perp}_1$, we must have $rc_j - td_j = 0$, since $g_1({F'}^{\perp}_1)$ is disjoint from $g_2(F_2)$, and similarly $g_2({F'}^{\perp}_2)$ is disjoint from $g_1(F_1)$. Therefore

$$rx - ty = \sum_i \sum_{j \in F^i_{1,k}} (rc_j - td_j) u(k,j)$$

Recall that for each coefficient the left hand side must be less than or equal to the right hand side. Evaluating both sides, we thus have that

$$
\begin{aligned}
&\sum_i \Big(\sum_{j \in F^i_{1,k}} (rc_j - td_j) u(k,j) \Big) \otimes f_2(e_i) \\
=&\sum_i \Big[\Big(\sum_{j \in F^i_{1,k}} (rc_j - td_j) u(k,j) \Big) \otimes \Big(\sum_l \sum_{m \in F^l_{2,i}} b(l,m) v(l,m) \Big) \Big] \\
=&\sum_i \Big[\sum_{j \in F^i_{1,k}} \sum_l \sum_{m \in F^l_{2,i}} (rc_j - td_j) b(l,m)\ u(k,j) \otimes v(l,m) \Big] \\
\leq&(1-t) \sum_i \Big[\Big(f_1(e_i) \wedge \sum_j u(k,j) \Big) \otimes \Big(\sum_l \sum_{n : s^n_l = m \in F^i_{2,l}} \mu_n v(l,m) \Big) \Big] \\
=&\sum_i (1-t) \Big[\Big(\sum_{j \in F^i_{1,k}} a(k,j) u(k,j) \Big) \otimes \Big(\sum_l \sum_{n : s^n_l = m \in F^i_{2,l}} \mu_n v(l,m) \Big) \Big] \\
=&\sum_i \Big[\sum_{j \in F^i_{1,k}} \sum_l \sum_{m \in F^i_{2,l}} (1-t) a(k,j) \Big(\sum_{n : s^n_l = m} \mu_n \Big) u(k,j) \otimes v(l,m) \Big]
\end{aligned}
$$

For each i, for all $j \in F^i_{1,k}$, for each l, and for all $m \in F^i_{2,l}$, the coefficient of $u(k,j) \otimes v(l,m)$ on the left hand side is $(rc_j - td_j)b(l,m)$, and on the right hand side, we have $(1-t)a(k,j) \sum_{n : s^n_l = m} \mu_n$. Thus

$$(rc_j - td_j) b(l,m) \leq (1-t) a(k,j) \sum_{n : s^n_l = m} \mu_n.$$

Let $A = A^{f_1}_E$ and $B = A^{f_2}_E$, as defined prior to Lemma 5.3.1. Adding across all $m \in F^i_{2,l}$, we have:

$$(rc_j - td_j) \sum_{m \in F^i_{2,l}} b(l,m) \leq (1-t) a(k,j) \sum_{m \in F^i_{2,l}} \sum_{n : s^n_l = m} \mu_n$$

so $(rc_j - td_j) B(l,i) \leq (1-t) a(k,j) \lambda^i_l$, where

$$\lambda^i_l = \sum_{m \in F^i_{2,l}} \sum_{n : s^n_l = m} \mu_n.$$

Observe that $\sum_i \lambda_l^i = 1$ for all rows l. Add up terms over all $j \in F_{1,k}^i$. Thus

$$\sum_{j\in F_{1,k}^i} (rc_j - td_j)B(l,i) \leq \sum_{j\in F_{1,k}^i} (1-t)a(k,j)\lambda_l^i \implies$$
$$(rC_i - tD_i)B(l,i) \leq (1-t)A(k,i)\lambda_l^i,$$

where $C_i = \sum_{j\in F_{1,k}^i} c_j$ and $D_i = \sum_{j\in F_{1,k}^i} d_j$. Now if

$$C' = \sum_{u(k,j)\in {F'}_1^{\perp}} c_j \quad \text{and} \quad D' = \sum_{u(k,j)\in {F'}_1^{\perp}} d_j,$$

then $\sum_i C_i + C' = 1$ and $\sum_i D_i + D' \leq 1$. Since $rx - ty \geq 0$, it follows that $rC_i - tD_i \geq 0$ and since for each j, $u(k,j) \in {F'}_1^{\perp}$ implies $rc_j - t_j d_j = 0$, we have $rC' - tD' = 0$. By Lemma 5.3.1, there exists a finite sequence $(\nu_l)_{l=1}^N$ such that $A(k) \leq C^2 \sum \nu_l B(l)$, with $\sum_l \nu_l = 1$ and $\nu_l \geq 0$. Then in particular,

$$(rC_i - tD_i)A(k,i) \leq C^2(1-t)A(k,i)\sum_l \nu_l \lambda_l^i$$

If $A(k,i) = 0$, then C_i and D_i are also 0, since $F_{1,k}^i$ is empty. Otherwise $A(k,i) > 0$, so for all i,

$$(rC_i - tD_i) \leq C^2(1-t)\sum_l \nu_l \lambda_l^i \implies$$
$$\sum_i (rC_i - tD_i) + rC' - tD' \leq C^2(1-t)\sum_l \sum_i \nu_l \lambda_l^i \implies$$
$$r - t \leq r - t(\sum_i D_i + D') \leq C^2(1-t)\sum_l \nu_l(\sum_i \lambda_l^i) \implies$$
$$r - t \leq C^2(1-t) \implies$$
$$r \leq C^2$$

Thus g_1 (and by similar argument g_2) is a C^2-embedding.

Finally, G itself has finitely many order extreme points, so by Theorem 2.5.4 it can be embedded into a $\ell_\infty^m(\ell_1^n)$ space, implying that $G \in \mathcal{K}'$. $\square$

Corollary 5.3.4. *Let E, F_1, F_2 be finite dimensional lattices, let $C \geq 1$, and suppose $f_1 : E \to F_1$ and $f_2 : E \to F_2$ are C- embeddings. Then for all $\varepsilon > 0$, there exist a lattice $G \in \mathcal{K}'$ and $(C + \varepsilon)$-embeddings $g_1 : F_1 \to G$ and $g_2 : F_2 \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

Proof. Pick δ such that $(1 + \delta)^2 C < C + \varepsilon$, and pick N such that there are $(1 + \delta)$-embeddings $\phi_j : F_j \to F_j' := \ell_\infty^N(\ell_1^{\dim F_j})$. Then each $\phi_j \circ f_j : E \to F_j'$ is a $C(1 + \delta)$-embedding. By Theorem 5.3.3, there exists $G \in \mathcal{K}'$ and $C(1+\delta)$-embeddings $g_j' : F_j' \to G$ for $j \in \{1, 2\}$ such that $g_1' \circ \phi_1 \circ f_1 = g_2' \circ \phi_2 \circ f_2$. Now let $g_j = g_j' \circ \phi_j$, and observe that each g_j is a $(1 + \delta)^2 C$-embedding, and $g_1 \circ f_1 = g_2 \circ f_2$. Since $(1 + \delta)^2 C < C + \varepsilon$, we are done. □

5.3.2 The Amalgamation Property for arbitrary Banach lattices

The above approach works well with finite dimensional lattices, but expanding to finitely generated lattices will lead to some additional complications since finitely generated lattices need not be finite dimensional. In fact, the separable isometrically universal lattice $\mathcal{U} = \mathcal{C}(\Delta, L_1(0, 1))$ can be generated by two elements (see Remark 3.1 in [51]). However, we can use this result to express separable lattices with sequences of finite dimensional lattices in order to demonstrate a general amalgamation.

Suppose now that E is a Banach lattice. Let α be a limit ordinal, and let $(E_\gamma)_{\gamma<\alpha}$ be a sequence of increasing sublattices of E such that $\overline{\cup_{\gamma<\alpha} E_\gamma} = E$. considering $(\gamma)_{\gamma<\alpha}$ as a net, define $\mathcal{E} \subseteq \prod E_\gamma$ by

$$\mathcal{E} = \{(x_\gamma)_{\alpha<\gamma} : \lim_\gamma x_\alpha = x \in E\}.$$

Essentially, $\mathcal{E}$ is a lattice of α-length sequences converging to elements in E, with norm $\|(x_\alpha)\|_{\mathcal{E}} = \sup_\alpha \|x_\alpha\|$.

Lemma 5.3.5. *Let E and $\mathcal{E}$ be as above, and let $\mathcal{E}_0$ be the ideal in $\mathcal{E}$ of null sequences. Then E is isometric to $\mathcal{E}/\mathcal{E}_0$.*

Proof. Let $x \in E$, and let $(x_\gamma)_\gamma \to x$, and let $[(x_\gamma)]$ denote the equivalence class induced by $\mathcal{E}_0$.

We will now show that the map $g : E \to \mathcal{E}/\mathcal{E}_0$ with $g(x) \mapsto [(x_\gamma)_\gamma]$ is an isometry.

First, it is well defined: if $(x_\gamma)_\gamma$ and $(y_\gamma)_\gamma$ converge to x, then $(y_\gamma - x_\gamma)_\gamma \in \mathcal{E}_0$, so $[(x_\gamma)_\gamma] = [(y_\gamma)_\gamma]$. By continuity of scalar multiplication and addition, $g(x)$ is linear. It also preserves norms. Note that $\|g(x)\|_{\mathcal{E}/\mathcal{E}_0} = \inf\{\|(x_\gamma)_\gamma\| : (x_\gamma)_\gamma \to x\}$, so $\|g(x)\|_{\mathcal{E}/\mathcal{E}_0} \geq \|x\|$, since $\|x_\gamma\| \to \|x\|$. For $\epsilon > 0$ and $(x_\gamma) \to x$, there exists some $\beta < \alpha$ such that for all $\gamma > \beta$, $\|x_\gamma - x\| < \epsilon$ consider then the α-sequence (x'_γ) with $x'_\gamma = 0$ for all $\gamma < \beta$ and $x'_\gamma = x_\gamma$ otherwise. Then $g(x) = [(x'_\gamma)_\gamma]$, and so $\|g(x)\|_{\mathcal{E}/\mathcal{E}_0} \leq \|x\| + \varepsilon$. In addition, the map is clearly surjective, since any $[(y_\gamma)] = g(x)$ where $(y_\gamma) \to x$. Thus g is a linear isometry.

Finally, g preserves lattice operations. First of all, g is positive. If $x \geq 0$, and $(x_\gamma)_\gamma \to x$, then $0 \leq (x_\gamma \vee 0)_\gamma \to x \vee 0 = x$. Since $\mathcal{E}_0$ is a lattice ideal, $g(x) = [(x_\gamma \vee 0)_\gamma] = [(x_\gamma)_\gamma] \vee [0] \geq 0$. In addition, $g(x)$ preserves disjointness: if $x \wedge y = 0$, then if $x_\gamma \to x$ and $y_\gamma \to y$, then $x_\gamma \wedge y_\gamma \to x \wedge y = 0$. Then

$$[(x_\gamma)] \wedge [(y_\gamma)] = [(x_\gamma \wedge y_\gamma)] = [0],$$

so g is a lattice homomorphism. Therefore g is a lattice isometry. □

Given a separable lattice E, by [51, Proposition 2.2], there exists a finitely branchable lattice E' such that $E \subseteq E' \subseteq E^{**}$. Let $(x_\sigma)_{T_{E'}}$ be the corresponding branching tree. Let $k_n \uparrow \infty$ where $k_n \in \mathbb{N}$ be a strictly increasing sequence, let $E'_{k_n} = \text{span}(x_\sigma : |\sigma| = k_n)$, and let $\mathcal{E}' \subseteq \prod_n E'_{k_n}$ be the lattice defined by

$$\mathcal{E}' = \{(x_i)_i : x_i \to x \in E'\},$$

with lattice norm $\|x\|_{\mathcal{E}'} = \sup_n \|x_n\|$. Finally, let $\mathcal{E}'_0 = \{x \in \mathcal{E}' : x_n \to 0\}$. By Lemma 5.3.5, $\mathcal{E}'/\mathcal{E}'_0$ is lattice isometric to E' itself. Furthermore, any finite dimensional lattice $F \in E$ can be approximated by a sublattice of some E_n for some n:

Lemma 5.3.6. *Let $E = \overline{\cup_n E_n}$ where (E_n) is an increasing sequence of lattices. Suppose $F \subseteq E$ is a finite dimensional sublattice. Then for all $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a $(1+\varepsilon)$-isometry $g : F \to E_n$ such that $\|g - Id|_F\| < \varepsilon$.*

Proof. Let $m = \dim F$, and recall from Section 3.3 the functions δ^m_k with $1 \leq k \leq m$, with $\delta^m_k(x_1, ..., x_m) = x_k - x_k \wedge (\vee_{i \neq k} x_i)$. Let $\vec{e} = (e_k)_k$ be the atoms of F. Now each δ^m_k is continuous, and for any m-length sequence $\vec{x}$ of positive elements, the elements $(\delta^m_k(\vec{x}))_k$ are mutually disjoint and positive. Thus since $\cup E_n$ is dense in X, for some n there exist corresponding positive $\vec{f} = (f_1, ..., f_m) \subseteq E_n$ such that for $1 \leq k \leq m$,

$\|\delta_k^m(\vec{e}) - \delta_k^m(\vec{f})\| < \varepsilon'/m$. Now since the e_k's are mutually disjoint, $\delta(\vec{e}) = e_k$. Let $g : F \to E_n$ be the lattice homomorphism generated by $g(e_k) = \delta_k^m(\vec{f})$. Then for any $\sum_k a_k e_k \in \mathbf{S}(F)$, we have

$$\left\| \sum e_k - \sum a_k g(e_k) \right\| \leq \sum_k^m |a_k| \|e_k - g(e_k)\| < \varepsilon'.$$

It follows that $1 - \varepsilon' < \|g(\sum_k^m a_k e_k)\| < 1 + \varepsilon'$, so g is a $\frac{1+\varepsilon'}{1-\varepsilon'}$-isometry. If we let $\frac{1+\varepsilon'}{1-\varepsilon'} < 1 + \varepsilon$, we have both that g is a $(1+\varepsilon)$-isometry and $\|Id|_F - g\| < \varepsilon$.

$\square$

We now state the following lemma:

Lemma 5.3.7. *Let E and A be a finitely branchable Banach lattices, and suppose $\phi : E \to A$ is an embedding. Let $(x_\sigma)_{\sigma \in T_E}$ and $(y_\sigma)_{\sigma \in T_A}$ be linearly dense spanning trees for E and A, respectively. Then for all $\varepsilon > 0$, there exist a strictly increasing sequence $(k_n)_n \subseteq N$ and $(1+\varepsilon)$-embedding $\phi' : \mathcal{E} \to \mathcal{A}$ generated by a sequence of maps $\phi_n : E_n \to A_{k_n}$ such that:*

1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\phi'} & \mathcal{A} \\ \big\downarrow{\scriptstyle q_E} & & \big\downarrow{\scriptstyle q_A} \\ E & \xrightarrow{\phi} & A \end{array}$$

2. *For each n, $\phi_n : E_n \to A_{k_n}$ is a $(1+\varepsilon/2^n)$-embedding.*

Proof. Let $\mathcal{E} \subseteq \prod_n E_n$. We will construct k_n as follows. Begin with $x_\emptyset \in E_+$, and suppose that $\|x_\emptyset\| = 1$. Pick $k_0 \in \mathbb{N}$ and $z_\emptyset \in \text{span}(\{y_\sigma : |\sigma| = k_0\})$ with $z_\emptyset \geq 0$ such that $\|z - \phi(x_\emptyset)\| < \varepsilon$. We then let $\phi_0'(x_\emptyset) = z_\emptyset$. For $n > 0$, since E_n is finite dimensional and embeds into A, by Lemma 5.3.6, pick k_n in such a way that such a way that there is a $\phi_n : E_n \to A_{k_n}$ with distortion level at most $(1+\varepsilon/2^n)$.

Let $\phi' = (\phi_n)_n$. Note that ϕ' sends atoms to disjoint elements and is a positive linear map. To show that property 1 is also fulfilled, we must first show that ϕ takes elements in $\mathcal{E}$ to elements in $\mathcal{A}$. Let $x \in \mathcal{E}$, with $(x_i) \to x' \in E$. Now $\phi(x_i) \in A$, and by continuity $\phi(x_i) \to \phi(x') \in A$ as well. Yet $\|\phi(x_i) - \phi_i(x_i)\| \leq \frac{\varepsilon}{2^i}$, so $\phi_i(x_i) \to \phi(x') \in \mathcal{A}$. In addition, if $x_i \to x \in E$, then $q_A \circ \phi'((x_i)_i) = \phi(x)$, which gives us commutativity, thus fulfilling property 1. $\square$

We are now ready to prove the following:

Theorem 5.3.8. *Let E, A_1, A_2 be separable Banach lattices, and let $f_1 : E \to A_1$ and $f_2 : E \to A_2$ be embeddings. Then there exists a separable Banach lattice G and embeddings $g_1 : A_1 \to G$ and $g_2 : A_2 \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*

Proof. Since each $f_i : E \to A_i$ is a lattice embedding for $i = 1, 2$, by [61, Theorem 1.4.19], each $f_i^{**} : E^{**} \to A_i^{**}$ is a lattice embedding. By Proposition 2.2 in [51], there exists a separable finitely branchable lattice $E \subset E' \subseteq E^{**}$ with a finitely branching tree $(x_\sigma)_{\sigma \in T_{E'}}$. Similarly, we can take the Banach lattice generated by $f_i^{**}(E')$ and A_i, and inject it into a finitely branchable A_i' with a corresponding finite branching tree $(y_\sigma)_{\sigma \in T_{A_i'}}$. Thus we can redefine f_1 and f_2 to be extended to E'.

Let $\varepsilon > 0$, and using Lemma 5.3.7, pick appropriate increasing sequences of natural numbers $k_n^1 \uparrow \infty$ and $k_n^2 \uparrow \infty$ generating $\mathcal{A}_1'$ and $\mathcal{A}_2'$ with accompanying $(1+\varepsilon)$-isometries f_1' and f_2' such that the following diagram commutes:

$$
\begin{tikzcd}
 & \mathcal{E}' \arrow[dl, "f_2'"'] \arrow[d, "q_{E'}"] \arrow[dr, "f_1'"] & \\
\mathcal{A}_2' \arrow[d, "q_{A_2'}"'] & E' \arrow[dl, "f_2"'] \arrow[dr, "f_1"] & \mathcal{A}_1' \arrow[d, "q_{A_1'}"] \\
A_2' & E \arrow[u, "Id"] \arrow[dl, "f_2"'] \arrow[dr, "f_1"] & A_1' \\
A_2 \arrow[u, "Id"] & & A_1 \arrow[u, "Id"']
\end{tikzcd}
$$

By the assumptions on Lemma 5.3.7, $f_j' = (\phi_n^j)_n$, where $\phi_n^j : E_n' \to A'_{j\ k_n^j}$ is a $(1+\varepsilon/2^n)$-isometry. Use Corollary 5.3.4 to get G_n and $(1+\varepsilon/2^{n-1})$-embeddings ψ_n^1 and ψ_n^2 such that $\psi_n^1 \circ \phi_n^1 = \psi_n^2 \circ \phi_n^2$, and let $g_1' = (\psi_n^1)_n$ and $g_2' = (\psi_n^2)_n$. Let $\mathcal{G}' \subseteq \prod_n G_n$ be the sublattice generated by $g_1'(\mathcal{A}_1')$ and $g_2'(\mathcal{A}_2')$, and equip $\mathcal{G}'$ with the sup-norm; that is, if $x \in \mathcal{G}'$, let $\|x\|_{\mathcal{G}'} = \sup \|x_n\|_{G_n}$. Now each g_j' is a $(1+2\varepsilon)$-embedding. Let $\mathcal{G}_0'$ be the ideal consisting of elements $x \in \mathcal{G}'$ such that $\|x_n\|_{G_n} \to 0$, and let $G = \mathcal{G}'/\mathcal{G}_0'$. Note that for each $j \in \{1, 2\}$, we have $g_j'(\mathcal{A}'_{j\ 0}) \subseteq \mathcal{G}_0'$. Thus g_j' induces well defined maps $g_j : A_j' \to G$, with $g_j = q_G \circ g_j' \circ q_{A_j'}^{-1}$. We therefore have the following commuting diagram:

$$\begin{tikzcd}
& \mathcal{E}' \arrow[rr, "f_1'"] \arrow[d, "q_{E'}"] \arrow[ddl, "f_2'"'] & & \mathcal{A}_1' \arrow[d, "q_{A_1'}"] \arrow[ddl, "g_1'"'] \\
& E' \arrow[rr, "f_1"] \arrow[ddl, "f_2"'] & & A_1' \arrow[ddl, "g_1"] \\
\mathcal{A}_2' \arrow[rr, "g_2'"] \arrow[d, "q_{A_2'}"'] & & \mathcal{G}' \arrow[d, "q_G"] & \\
A_2' \arrow[rr, "g_2"] & & G &
\end{tikzcd}$$

It remains to show that each g_j is in fact an embedding. To this end, we note that if $z \in G$, then $\|z\| = \inf\{\|y\| : q_G(y) = z\}$. Let $x \in A_1'$. Pick $y \in \mathcal{A}_1'$ with $\|y\| < 1+\delta$ such that $y_i \to x$. This can be done by picking n such that for all $n \geq N$ $\|x - y_n\| < \delta$, $\varepsilon/2^{n-1} < \delta$, and furthermore, we can assume that for all $n < N$, $y_n = 0$. It then follows that $\frac{1}{(1+\delta)^2} \leq \|g_1'(y)\|_{\mathcal{G}'} \leq (1+\delta)^2$, so $\|q_G g_1'(y)\| \leq (1+\delta)^2$, Thus $\|g_1(x)\|_{G'} \leq (1+\delta)^2$. In addition, for any $z \in \mathcal{G}_0'$, since for all $\delta' > 0$ we have $\|z_n\|_{G_n} \leq \delta'$ for all large enough n, it follows that $\|z - g_1'(y)\| > \frac{1}{(1+\delta)^2} - \delta'$. Thus $\|g_1(x)\|_G \geq \frac{1}{(1+\delta)^2}$. δ can be chosen to be arbitrarily small, so $\|g_1(x)\|_G = 1$.

Finally, we show that g_j preserves disjointness and is a positive map. Let $x \in A_{j+}'$, and chose a sequence $y = (y_i)_i \in \mathcal{A}_{j+}'$ with $y_i \to x$. Then $g_j'(y) \geq 0$, so $q_G g_j'(y) = g_j(x) \geq 0$. To show preservation of disjointness, let $x, x' \geq 0$ be disjoint elements, and let $y = (y_i)_i \in q_{\mathcal{A}_j'}^{-1}(x)$ and similarly let $y' = (y_i')_i \in q_{\mathcal{A}_j'}^{-1}(x')$. Then $y \wedge y' \in \mathcal{A}_{j\ 0}'$; since $(y_i)_i \to x$ and $(y_i')_i \to x'$, we have

$$y \wedge y' = (y_i \wedge y_i')_i \to x \wedge x' = 0,$$

so $g_j'(y) \wedge g_j'(y') = g_j'(y \wedge y') \in \mathcal{G}_0'$, which means that $g_j(x) \wedge g_j(x') = q_G g_j'(y) \wedge q_G g_j'(y') = q_G g_j'(y \wedge y') = 0$. Thus g_j is an embedding.

To show separability, we simply restrict g_1 and g_2 to A_1 and A_2, and replace G with the lattice generated by $g_1(A_1) \bigcup g_2(A_2)$. Thus if A_1 and A_2 are both separable, then so is G. $\square$

Remark 5.3.9. We can also ensure that G is finitely generated, since we can embed G into $\mathcal{U}$ if necessary. Thus $\mathcal{K}$ has the AP.

We can expand Theorem 5.3.8 for arbitrary lattices with a similar proof.

Theorem 5.3.10. *Let E, F_1, F_2 be Banach lattices, and let $f_i : E \to F_i$, with $i \in \{1, 2\}$ be embeddings. Then there exists a lattice G and isometric embeddings $g_i : F_i \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$. Furthermore, if F_i has density character no more than κ, we can ensure that G does as well.*

Proof. We prove this by ordinal induction over the density character κ. For the base case of $\kappa = \aleph_0$, this was already proven in Theorem 5.3.8. Suppose now that we have shown the same for all lattices of density character less than κ. Let $(z_\gamma)_{\gamma<\kappa}$ be a κ-sequence dense in E, and let $(x^i_\alpha)_{\alpha<\kappa}$ be κ-length sequences dense in F_i. Let $E^\beta = BL((z_\alpha)_{\alpha<\beta})$ and let $F^\beta_i = BL((x^i_\alpha)_{\alpha<\beta} \cup f_i(E^\beta))$. Then $E^\beta \uparrow E$, $F^\beta \uparrow F$, and $f_i(E^\beta) \subseteq F^\beta_i$. Now each f_i induces an embedding $\phi_i : \mathcal{E} \to \mathcal{F}_i$, where $\phi_i((y_\beta)_{\beta<\kappa}) = (f_i(y_\beta))_{\beta<\kappa}$.

Both E^β and the F^β_i's have dense subsets of size strictly less than κ, so by induction, pick G^β and embeddings $\psi^\beta_i : F^\beta_i \to G^\beta$ such that $\psi^\beta_1 \circ f_1|_{E^\beta} = \psi^\beta_2 \circ f_2|_{E^\beta}$. Let $\psi_i = (\psi^\beta_i)_{\beta<\alpha}$, and let $\mathcal{G}$ be the sublattice of $\prod_\beta G^\beta$ generated by the elements of $\psi_i(\mathcal{F}_i)$. Let $\mathcal{G}_0$ be the ideal in $\mathcal{G}$ of nets converging in norm to 0, and let $G = \mathcal{G}/\mathcal{G}_0$. Now let $g_i = q_G \circ \psi_i \circ q_E^{-1}$. Use the same argument as in Theorem 5.3.8 to show that each g_i is well defined, an embedding, and together with G give the desired amalgamation. Finally, G has the desired density character if we restrict it to the lattice generated by $g_1(F_1) \cup g_2(F_2)$. □

We end this section with some additional results on the interplay between the AP and C-embeddings. In each of these cases, we can perturb lattices or maps that change C-embeddings into embeddings in exchange for full commutativity or preservation of the original norm:

Theorem 5.3.11. *Let $f : A \to X$ be a C- embedding. Then there exists a C-equivalent renorming $|||\cdot|||$ of X such that $f : A \to (X, |||\cdot|||)$ is an embedding. Furthermore,*

- *if f is an expansion (that is, if f^{-1} is contractive), then we can make $|||\cdot||| \leq \|\cdot\|$.*
- *if f is a contraction, then we can make $|||\cdot||| \geq \|\cdot\|$.*
- *if A and X are both in $\mathcal{K}'$, then we can ensure that $(X, |||\cdot|||)$ is also in $\mathcal{K}'$.*

Proof. We start with a proof for the case when f is an expansion. Let

$$\mathbf{B}' = CSCH\big(f(\mathbf{B}(A)) \cup \mathbf{B}(X)\big)$$

be the unit ball of $|||\cdot|||$. Observe that $\mathbf{B}' \supseteq \mathbf{B}(X)$ and $f(\mathbf{B}(A)) \subseteq C\mathbf{B}(X)$, so $\frac{1}{C}\|\cdot\| \leq |||\cdot||| \leq \|\cdot\|$.

We now show that $f : A \to (X, |||\cdot|||)$ is an embedding. Suppose that there exist $z_n \leq t_n f(x_n) + (1 - t_n)y_n$ with $(1 + \alpha)f(x) = \lim_n z_n$, with $x, x_n, y_n \geq 0$, $\alpha \geq 0$, $\|x_n\|, \|y_n\| \leq 1$, $0 \leq t_n \leq 1$, and $\|x\| = 1$. By compactness, we can suppose t_n converges to t, and just let $z_n \leq tf(x_n) + (1-t)y_n$. Furthermore, we can assume that $\|f((1+\alpha)x - tx_n)\| \to b_x$. Then for all n, we have

$$f((1 + \alpha)x - tx_n) \leq (1 - t)y_n + \delta_n$$

with $\|\delta_n\| \to 0$. We then have

$$1 + \alpha - t \leq \|f((1 + \alpha)x - tx_n)\| \leq (1 - t) + \|\delta_n\|.$$

Thus $1 + \alpha - t \leq b_x \leq 1 - t$, so $\alpha = 0$.

For contractive f, let $\mathbf{B}'$ be the closed solid convex hull of $f(\mathbf{B}(A)) \cup \frac{1}{C}\mathbf{B}(X)$. Note here that $\frac{1}{C}\mathbf{B}(X) \subseteq \mathbf{B}' \subseteq \mathbf{B}(X)$, so $\|\cdot\| \leq |||\cdot||| \leq C\|\cdot\|$. Then use the same type of argument.

For the general case, observe that Cf is an expansion which is also a C^2-embedding. Then by the proof of the first case, there is C^2-equivalent renorming $|||\cdot||| \leq \|\cdot\|$ of X with $Cf : A \to (X, |||\cdot|||)$ an embedding. Now take the new norm of X to be $C|||\cdot|||$. Then $f : A \to (X, C|||\cdot|||)$ is an embedding, and $C|||\cdot|||$ is C-equivalent to $\|\cdot\|$.

Finally, if $A, X \in \mathcal{K}'$, the unit ball of the renormed lattice $(X, |||\cdot|||)$ has finitely many order extreme points, so by Corollary 5.3.2, $(X, |||\cdot|||)$ is also in $\mathcal{K}'$. $\square$

Theorem 5.3.11 can be used to generalize Theorem 5.3.10 to diagrams involving C-isometries:

Corollary 5.3.12. *Let $f_i : E \to F_i$ with $i = 1, 2$ be C_i-embeddings for lattices E, F_1, and F_2. Then:*

- *There exist a lattice G and C_i-embeddings $g_i : F_i \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*
- *There exist a lattice G, an embedding $g_1 : F_1 \to G$, and a C_1C_2-embedding $g_2 : F_2 \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$.*
- *If E, F_1, and F_2 are in $\mathcal{K}'$, we can ensure $G \in \mathcal{K}'$ as well.*

Proof. For the first part, let $F_i' = (F_i, |||\cdot|||)$ be C_i-equivalent renormings such that f_i is an embedding into F_i'. By Theorem 5.3.10 (Theorem 5.3.3), there exists G and embeddings $g_i : F_i' \to G$ such that $g_1 \circ f_1 = g_2 \circ f_2$. Since F_i' is C_i-equivalent to F_i, each g_i is a C_i-embedding on F_i. For the second part, use Theorem 5.3.11 to simply renorm G with a C_1-equivalent norm $|||\cdot|||$ so that $g_1 : F_1 \to (G, |||\cdot|||)$ is now an embedding. Then $g_2 : F_2 \to (G, |||\cdot|||)$ is a C_1C_2-embedding. For both parts, G can be in $\mathcal{K}'$ if E, F_1, and F_2 are in $\mathcal{K}'$. □

Theorem 5.3.13. *Suppose $f : X \to Y$ is a $(1+\varepsilon)$-embedding, and suppose X, Y are in $\mathcal{K}$ (or $\mathcal{K}'$). Then there exists a lattice $Z \in \mathcal{K}$ ($\mathcal{K}'$) and embeddings $g : X \to Z$ and $h : Y \to Z$ such that $\|g - h \circ f\| \leq \varepsilon$.*

Proof. Let $j_1 : X \to X \oplus_\infty f(X)$, with $j_1(x) = x \oplus \frac{1}{1+\varepsilon} f(x)$. Let $j_2 : f(X) \to X \oplus_\infty f(X)$ with $j_2(f(x)) = \frac{1}{1+\varepsilon} x \oplus f(x)$. Note then that since $\frac{1}{1+\varepsilon}\|f(x)\| \leq \|x\| \leq (1+\varepsilon)\|f(x)\|$, j_1 and j_2 are both embeddings. Then

$$\begin{aligned}\|j_1(x) - j_2 f(x)\| =& \left\| \left(1 - \frac{1}{1+\varepsilon}\right) x \oplus \left(\frac{1}{1+\varepsilon} - 1\right) f(x) \right\| \\ =& \frac{\varepsilon}{1+\varepsilon} \|x \oplus -f(x)\| \leq \varepsilon \|x\|.\end{aligned}$$

If f is surjective, then let $g = j_1$ and $h = j_2$, and we are done. Otherwise, $f(X) \subseteq Y$ and $j_2 : f(X) \to X \oplus_\infty f(X)$ in an embedding, so use Theorem 5.3.10 (or Theorem 5.3.3) to get a lattice Z in $\mathcal{K}$ (respectively $\mathcal{K}'$) and embeddings $h_1 : Y \to Z$ and $h_2 : X \oplus_\infty f(X) \to Z$ such that $h_1|_{f(X)} = h_2 \circ j_2$. Then for all $x \in X$,

$$\|h_2 j_1(x) - h_1 f(x)\| = \|h_2 j_1(x) - h_2 j_2 f(x)\| = \|j_1(x) - h_2(f(x))\| \leq \varepsilon \|x\|.$$

Let $g = h_2 \circ j_1$ and $h = h_1$, and we are done. □

Corollary 5.3.14. *Let E, F_1, F_2 be lattices in $\mathcal{K}$ (or $\mathcal{K}'$), and let $f_j : E \to F_j$ be $(1+\varepsilon)$-embeddings. Then there exist $H \in \mathcal{K}$ ($\mathcal{K}'$) and embeddings $g_j : F_j \to G$ such that $\|g_1 \circ f_1 - g_2 \circ f_2\| \leq 2\varepsilon$.*

Proof. By Theorem 5.3.13 there exist $F_j' \in \mathcal{K}$ ($\mathcal{K}'$) and embeddings $f_j' : E \to F_j'$ and $\phi_j : F_j \to F_j'$ such that $\|f_j' - \phi_j \circ f_j\| \leq \varepsilon$. Now use Theorem 5.3.10 (or Theorem 5.3.3) to get $H \in \mathcal{K}$ ($\mathcal{K}'$) and embeddings $g_j' : F_j' \to H$ with $g_1' \circ f_1' = g_2' \circ f_2'$. Let $g_j = g_j' \circ \phi_j$.

Then

$$\begin{aligned}\|g_1 \circ f_1 - g_2 \circ f_2\| =& \|g_1' \circ \phi_1 \circ f_1 - g_2' \circ \phi_2 \circ f_2\| \\ \leq& \|g_1' \circ (\phi_1 \circ f_1 - f_1')\| + \|g_2' \circ (\phi_2 \circ f_2 - f_2')\| \leq 2\varepsilon.\end{aligned}$$

$\square$

5.4 The approximately ultra-homogeneous separable lattice $\mathfrak{BL}$

The main result of this section is:

Theorem 5.4.1. *The class $\mathcal{K}$ of finitely generated separable Banach lattices is a Fraïssé class. Thus there exists a separable approximately ultra-homogeneous Banach lattice $\mathfrak{BL}$.*

The level of homogeneity in $\mathfrak{BL}$ cannot significantly be strengthened. For one, $\mathfrak{BL}$ can only be "approximately" ultra-homogeneous, since no lattice automorphism can map non-weak units to weak units. $\mathfrak{BL}$ is clearly also atomless. If there were an atom e in $\mathfrak{BL}$, any automorphism would have to map it to another atom. Thus two embeddings $g_1 : \mathbb{R} \to < e > \subseteq \mathfrak{BL}$ and $g_2 : \mathbb{R} \to < x > \subseteq \mathfrak{BL}$ such that x both is disjoint from e and not an atom cannot be arbitrarily approximated by an automorphism.

$\mathfrak{BL}$ is isometrically universal for separable Banach lattices, but it is not isometric to $\mathcal{U}$ because the latter is not approximately ultra-homogeneous and Fraïssé limits are unique up to isometry. Indeed, let $< e >$ be a one-dimensional lattice generated by e, let $f_1(e) = a := \vec{1}_\Delta \otimes \chi_{[0,1]}$, $f_2(e) = b := \vec{1}_K \otimes \chi_{[0,1]}$, where $K \subseteq \Delta$ is a proper clopen subset, and let $b' = a - b$. Let ϕ be any automorphism over $\mathcal{U}$. Now the sets $K_b = \{k \in \Delta : \|\phi(b)(k)\|_1 = 1\}$ and $K_{b'} = \{k \in \Delta : \|\phi(b')(k)\|_1 = 1\}$ are non-empty, and furthermore $b(K_{b'}) = 0$, and vice versa, since $\phi(< b, b' >)$ is isometric to ℓ^2_∞. It follows that for $k \in K_{b'}$, we have $\|\phi(b)(k) - a(k)\|_1 = 1$, so $\|\phi(b) - a\| \geq 1$.

Proof of theorem. It is clear that $\mathcal{K}$ has the HP and the JEP. It also has the CP by virtue of the fact that each function symbol in the language of Banach lattices has a fixed modulus of continuity independent of its interpretation. By Theorem 5.3.8, it has the AP. It remains to show that it has the PP. We need to show that the class of finitely generated Banach lattices is both separable and complete under the metric $d^{\mathcal{K}}$. For separability, let $(x^n)_n$ be a countable dense subset of $\mathcal{U}$. Then the set $\{< x_{i_1}, ..., x_{i_n} >\}$ of lattices generated by finitely many elements in $(x_n)_n$ is itself a countable dense subset of $\mathcal{K}_n$.

To show completeness, we use Theorem 5.3.8 and the fact that Banach lattices are closed under direct limits. Let $(\overline{a_i})_i$ be a Cauchy sequence of tuples generating structures in $\mathcal{K}_n$. By passing to a subsequence if necessary, we can assume that $d^{\mathcal{K}}(\overline{a_i}, \overline{a_{i+1}}) < \frac{1}{2^{i+1}}$. For $\overline{a_i}$ and $\overline{a_{i+1}}$, let B_i^1 be a finitely generated lattice containing isometric copies of $< \overline{a_i} >$ and $< \overline{a_{i+1}} >$ such that $d(\overline{a_i}, \overline{a_{i+1}}) < d^{\mathcal{K}}(\overline{a_i}, \overline{a_{i+1}}) + \frac{1}{2^{i+1}}$. Note then for each i, we have embeddings $< \overline{a_{i+1}} > \to B_i^1, B_{i+1}^1$, so use amalgamation to embed B_i^1 and B_{i+1}^1 into some finitely generated space B_i^2 where the associated diagram commutes. Proceed inductively in a similar manner: each B_{i+1}^k injects into B_i^{k+1} and B_{i+1}^{k+1}, so use amalgamation to inject them into some finitely generated B_i^{k+2}. The resulting commutative diagram illustrates the process:

$$\begin{tikzcd}
< \overline{a_1} > \arrow[r] & B_1^1 \arrow[r] & B_1^2 \arrow[r] & B_1^3 \arrow[r] & \dots \\
< \overline{a_2} > \arrow[r] \arrow[ur] & B_2^1 \arrow[r] \arrow[ur] & B_2^2 \arrow[ur] & & \\
< \overline{a_3} > \arrow[r] \arrow[ur] & B_3^1 \arrow[ur] & & & \\
\vdots \arrow[ur] & & & &
\end{tikzcd}$$

Let X be the closed inductive limit of the sequence of lattices $(B_1^n)_n$. X is itself separable, though it need not be finitely generated. It also contains an isometric copy of each $< \overline{a_i} >$ and for each $\overline{a_i}, \overline{a_j} \subseteq X$ with $i \leq j$, we have

$$d(\overline{a_i}, \overline{a_j}) < \sum_{k=i}^{j-1} \left(d^{\mathcal{K}}(\overline{a_k}, \overline{a_{k+1}}) + \frac{1}{2^{k+1}} \right) < \sum_{k=i}^{j-1} 2^{-k} < 2^{-i+1.}$$

Thus $(\overline{a_i})_i$, as a sequence of tuples in X, is Cauchy. Let $\overline{a} = \lim_i \overline{a_i}$. Since X is complete, the sublattice $< \overline{a} >$ exists, which implies the completion of the metric $d^{\mathcal{K}}$. Thus $\mathcal{K}$ has the PP, and we are done. $\square$

We continue with an additional characterization of $\mathfrak{BL}$. In particular, $\mathfrak{BL}$ is finitely branchable and finitely generated. To this end, we concentrate on the sub-class $\mathcal{K}'$.

The $\ell_\infty^m(\ell_1^n)$ lattices are in certain ways analogues of ℓ_∞^n spaces. For one, recall the definition of an injective Banach space E: If $T : F \to E$ is a linear map and F is a subpace of G, then there exists a linear map $\hat{T} : G \to E$ extending T such that $\|T\| = \|\hat{T}\|$. There is also a lattice analogue of injectivity: We say E is an injective lattice if for all lattices $F \subseteq G$ and any positive linear maps $T : F \to E$, then there exists a positive linear map $\hat{T} : G \to E$ extending T such that $\|T\| = \|\hat{T}\|$. The injective finite dimensional Banach spaces are exactly the ℓ_∞^n spaces. By [19, Theorem 5.2], the ℓ_∞-sums of finite dimensional ℓ_1 spaces make up the collection of finite dimensional injective lattices. Furthermore, the Gurarij space in particular can be constructed as an inductive limit of ℓ_∞^n Banach spaces. We will now also show that $\mathfrak{BL}$ can be constructed as an inductive limit of $\ell_\infty^m(\ell_1^n)$ lattices.

Lemma 5.4.2. *$\mathcal{K}'$ is an incomplete Fraïssé class that is dense in $\mathcal{K}$. In particular, the Fraïssé metric $d^{\mathcal{K}'}$ isometrically coincides with $d^{\mathcal{K}}$.*

Proof. $\mathcal{K}'$ has the HP by its definition. By Theorem 5.3.3, it has the AP. Clearly it also has the JEP: given two $X, Y \in \mathcal{K}'$, we also have $A \oplus_\infty B \in \mathcal{K}'$. To show density in $\mathcal{K}$, let $< \overline{a} >$ be a finitely generated lattice, and embed $< \overline{a} >$ into $\mathcal{U}$. Let $(x_\sigma)_{\sigma \in T}$ be the finitely branching tree comprised of elements in $\mathcal{U}$ of the form $\chi_{N_\sigma} \otimes \chi_{Q_k}$ where Q_k is a diadic interval of length 2^{-n}, $|\sigma| = n$, and $N_\sigma \subseteq \Delta = \{0,1\}^{\mathbb{N}}$ is the set consisting of all infinite branches starting with σ. Now $\mathrm{span}((x_\sigma)_{\sigma \in T})$ is dense in $\mathcal{U}$. Let $S_n := \{x_\sigma : |\sigma| = n\}$, and observe that $\mathrm{span}\{S_n\}$ is itself a $\ell_\infty^{2^n}(\ell_1^{2^n})$ space. Given $\varepsilon > 0$, choose n and $\overline{x} \subseteq \mathrm{span}(S_n)$ such that $d(\overline{a}, \overline{x}) < \varepsilon$. Then the lattice $< \overline{x} > \in \mathcal{K}'$ is sufficiently close to $< \overline{a} >$ in $d^{\mathcal{K}}$.

To show separability of $\mathcal{K}'$ and the CCP, it is sufficient to show that $d^{\mathcal{K}}|_{\mathcal{K}'} = d^{\mathcal{K}'}$. Clearly $d^{\mathcal{K}}|_{\mathcal{K}'} \leq d^{\mathcal{K}'}$, so we need only to show the opposite inequality. Let $d^{\mathcal{K}}(\overline{a}, \overline{b}) = \delta$, and let $\varepsilon > 0$. Choose embeddings ϕ_A and ϕ_B from $< \overline{a} >$ and $< \overline{b} >$ into $\mathcal{U}$ such that $d(\phi_A(\overline{a}), \phi_B(\overline{b})) < \delta + \varepsilon$. By Lemma 5.3.6, given $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $(1+\varepsilon)$-embeddings $f_A : A \to D := \mathrm{span}(S_n)$ and $f_B : B \to D$ such that $\|f_A - \phi_A\| < \varepsilon$ and $\|f_B - \phi_B\| < \varepsilon$. Now note that $d(f_A(\overline{a}), f_B(\overline{b})) < \delta + 3\varepsilon$. Use Theorem 5.3.11 to renorm D with a $(1+\varepsilon)$-equivalent renorming $|\!|\!|\cdot|\!|\!|$ so that $f_B : B \to D' = (D, |\!|\!|\cdot|\!|\!|) \in \mathcal{K}'$ is an embedding. Then f_A is a $(1+\varepsilon)^2$-embedding into D'. Finally, use Theorem 5.3.13 to get some $C \in \mathcal{K}'$ and embeddings $g_A : A \to C$ and $g_{D'} : D' \to C$ such that

$\|g_{D'} \circ f_A - g_A\| \leq 2\varepsilon + \varepsilon^2$. Then for each i, we have

$$\begin{aligned}
\|g_A(a_i) - g_{D'} f_B(b_i)\|_C &\leq \|g_A(a_i) - g_{D'} f_A(a_i)\|_C + \|g_{D'} f_A(a_i) - g_{D'} f_B(b_i)\|_C \\
&\leq 2\varepsilon + \varepsilon^2 + \|f_A(a_i) - f_B(b_i)\|_{D'} \\
&\leq 2\varepsilon + \varepsilon^2 + (1+\varepsilon) d(f_A(\overline{a}), f_B(\overline{b})) \\
&\leq 2\varepsilon + \varepsilon^2 + (1+\varepsilon)(\delta + 3\varepsilon).
\end{aligned}$$

We can let ε get arbitrarily small, so $d^{\mathcal{K}'} \leq d^{\mathcal{K}}|_{\mathcal{K}'}$, and we are done. □

It is known that incomplete Fraïssé classes admit a Fraïssé limit for the completion of the class, but here we will explicitly show that the construction of the limit $\mathfrak{BL}$ need only involve an increasing sequence of lattices in $\mathcal{K}'$. In order to prove the following theorem, we use approximate isometries as described in [13], that is, bi-Katetov maps $\psi : X \times Y \to [0, \infty]$ with X and Y both metric spaces. Recall that ψ is bi-Katetov if for all $x, x_0 \in X$ and $y, y_0 \in Y$, $|\psi(x,y) - d(x,x_0)| \leq \psi(x_0, y)$ and $|\psi(x,y) - d(y,y_0)| \leq \psi(x,y_0)$. In this context, approximate isometries provide information about how generating tuples $\overline{a}$ and $\overline{b}$ relate in ambient spaces.

Approximate isometries can be induced by finite partial embeddings, i.e., partial functions $f : X \rightharpoonup Y$, with $\mathrm{dom}(f) = X_0$ a finite set, which induce lattice embeddings $f :< X_0 > \to Y$. More generally, for any $X_0 \subseteq X$ and a (not necessarily finite) partial embedding $f : X_0 \to Y$, we let $\psi_f : X \times Y \to \mathbb{R}$ be defined by $\psi_f(x,y) = \inf_{z \in X_0} \|x - z\| + \|y - f(z)\|$. Observe that if $x \in X_0$, then $\psi_f(x,y) = \|f(x) - y\|$. If $X_0 \subseteq X$, we also have an approximate isometry $\psi_{Id_{X_0}} : X_0 \times X \to \mathbb{R}$ with Id_{X_0} the inclusion maps from X_0 to X, where $\psi_{Id_{X_0}}(x,y) = \|x - y\|$.

There is also a "pseudoinverse" operation: if $\psi(x,y)$ is an approximate isometry, we let $\psi^*(y,x) = \psi(x,y)$. Clearly $\psi^{**} = \psi$. We can also "compose" approximate isometries. If $\phi : X \times Y \to \mathbb{R}$ and $\psi : Y \times Z \to \mathbb{R}$ are approximate isometries, then $\psi\phi : X \times Z \to \mathbb{R}$ with $\psi\phi(x,z) = \inf_{y \in Y}(\phi(x,y) + \psi(y,z))$ is also an approximate isometry by [13, Lemma 2.3(i)]. For example, if $f : A_0 \to C$ and $g : B_0 \to C$ generate embeddings from $< A_0 >$ and $< B_0 >$ to C respectively, then the map $\psi_g^* \psi_f :< A_0 > \times < B_0 > \to \mathbb{R}$, where

$$\psi_g^* \psi_f(x,y) = \inf_{z \in C}(\psi_f(x,z) + \psi_g(y,z)).$$

is also an approximate isometry. Note that if $A_0 = \overline{a}$, $B_0 = \overline{b}$, and $d(f(\overline{a}), g(\overline{b}))$

is small, then $\psi_g^*\psi_f(a_i, b_i)$ will also be small, and the converse holds true as well. In fact, for $x \in A_0$ and $y \in B_0$, we have $\psi_g^*\psi_f(x, y) = \|f(x) - g(y)\|$ (here A_0 and B_0 need not be finite). Thus we can see approximate isometries as marking conditions for the "strength" of a joint embedding. An approximate isometry ψ may originally be defined on some $X_0 \times Y_0$, with $X_0 \subseteq X$ and $Y_0 \subseteq Y$, but it can be extended to $X \times Y$ by the composition $\psi_{Id_{Y_0}}\psi\psi^*_{Id_{X_0}}$. However, if the ambient spaces are clear from context, we will just write ψ to refer to the extended approximate isometry. Finally, composition and involution as described above work analogously together like the multiplication and inversion group operations. In particular, composition is associative and $(\psi\phi)^* = \phi^*\psi^*$ (see [13, Lemma 2.3(ii)]).

We say that ψ is **refined by**, or **coarsens** ϕ if $\phi(x, y) \leq \psi(x, y)$ for all $(x, y) \in X \times Y$. Given lattices X and Y, we let $\mathcal{A}px(X, Y) \subseteq [0, \infty]^{X\times Y}$, equipped with the product topology on $[0, \infty]^{X\times Y}$, be the set of all approximate isometries generated by finite partial embeddings between elements in $\mathcal{K}'$, composition, coarsening, and any pointwise limit of such maps. For $\psi \in \mathcal{A}px(X, Y)$, we let $\mathcal{A}px^{<\psi}(X, Y)$ be the interior of the set of refinements of ψ. If $\mathcal{A}px^{<\psi}(X, Y) \neq \emptyset$, we say that ψ is a **strictly approximate isometry** and use the notation $\phi < \psi$ to mean $\phi \in \mathcal{A}px^{<\psi}(X, Y)$. Intuitively, strictly approximate isometries do not impose strong conditions on possible joint embeddings except on some finite set (see Lemma [14, Lemma 3.8(ii)]), so they leave much room for refinement. While the set $\mathcal{A}px(X, Y)$ seems complicated, by [13, Lemma 3.8(iv)], it actually is comprised of the closure of coarsening and pointwise limits of approximate isometries in the form of $\psi_g^*\psi_f$ (extended to $X \times Y$) where f and g are finite partial embeddings.

Suppose $\psi : X \times Y \to \mathbb{R}$ is an approximate isometry, and let $r > 0$. We say that ψ is r-total if $\psi^*\psi \leq \psi_{Id_X} + 2r$. It is not hard to show that if $f : X \to Y$ is an embedding, then $\psi_f^*\psi_f = \psi_{Id_X}$, so any such ψ_f is r-total for all $r > 0$.

Approximate isometries can also be used to characterize the AP: for every $A, B \in \mathcal{K}'$, $< \overline{a} >\subseteq A$ and embedding $f :< \overline{a} >\to B$, there exist $C \in \mathcal{K}'$ and embeddings $g : A \to C$ and $h : B \to C$ such that $\psi_h^*\psi_g \leq \psi_f$ (with the necessary extensions on f). See [14, Definition 3.5(iii)] for the generalized definition of the NAP using approximate isometries. Using the fact that $\psi_h^*\psi_g(x, y) = \|g(x) - h(y)\|$ and $\psi_f(x, y) = \|f(x) - y\|$, one can easily show that this definition is equivalent to our current working definition of the AP. Note also the inequality; this is due to the fact that f is only defined on $< \overline{a} >$ while g is defined on all of A, so the extension of ψ_f to $A \times B$ contains less limiting information than $\psi_h^* \psi_g(x, y)$.

We are now ready to prove the following:

Theorem 5.4.3. *$\mathfrak{BL}$ can be constructed as the limit of an increasing sequence of $\ell_\infty^m(\ell_1^n)$ lattices. In particular, it is finitely branchable.*

Proof. We construct an increasing sequence of finite dimensional lattices $\mathcal{A}_n$ as in the proof of Lemma 3.17 in [13] with $\mathfrak{BL}$ isometric to $\overline{\bigcup_n \mathcal{A}_n}$. Let $\mathcal{A}_1 \in \mathcal{K}'$, and let $K_{n,0}$ be a countable dense subset of $\mathcal{K}'_n$. Since $\mathcal{K}'$ is dense in $\mathcal{K}$, we have $K_{n,0}$ dense in $\mathcal{K}_n$. We proceed by induction. Suppose $\mathcal{A}_k$ has been defined for all $k \leq n$. Suppose also that $A_{k,0} \subseteq \mathcal{A}_k$ is countable and dense in $\mathcal{A}_k$ for all $k \leq n$, with $A_{k,0} \subseteq A_{k+1,0}$ for each $k < n$.

By [13, Lemma 3.8(ii)], for any finite tuples $\overline{a}$ and $\overline{b}$ we can ensure the existence of a countable set $C(\overline{a}, \overline{b}) \subseteq \mathcal{K}'$ such that every $C \in C(\overline{a}, \overline{b})$ contains an isometric copy of $< \overline{a} >$ and $< \overline{b} >$, and every strictly approximate isometry $\psi : \overline{b} \times \overline{a} \to \mathbb{Q}$ can be refined in $< \overline{b} > \times < \overline{a} >$ by some $\psi_f^* \psi_g$ with $f :< \overline{a} > \to C$ and $g :< \overline{b} > \to C$ for some $C \in C(\overline{a}, \overline{b})$ (by [13, Lemma 2.8(ii)], such strictly approximate isometries $\psi : \overline{b} \times \overline{a} \to \mathbb{Q}$ actually exist). Let

$$C_k = \bigcup_{\substack{\overline{b} \in K_{n,0} \\ \overline{a} \subseteq A_{k,0}}} C(\overline{a}, \overline{b}).$$

To construct $\mathcal{A}_{n+1}$, take the first n lattices $C_k^1, ..., C_k^n$ in each C_k for $k \leq n$, and amalgamate them one after another. Here $\overline{a_{i,j}} \subseteq A_{i,0}$ for some $i \leq n$, and $< \overline{a_{i,j}} >$ and $< \overline{b_{i,j}} >$ both into $C_i^j \in C(\overline{a_{i,j}}, \overline{b_{i,j}}) \subseteq C_i$:

$$\begin{tikzcd}
\mathcal{A}_n \arrow[rr] & & * \dots & * \arrow[r, "\iota'"] & * & \dots & * \arrow[r] & \mathcal{A}_{n+1} \\
< \overline{a_{1,1}} > \arrow[u] \arrow[r] & C_1^1 \arrow[ur] & < \overline{a_{i,j}} > \arrow[u] \arrow[r, "f"] & C_i^j \arrow[ur, "h"] & \dots & < \overline{a_{n,n}} > \arrow[u] \arrow[rr] & & C_n^n \arrow[u] \\
< \overline{b_{1,1}} > \arrow[ur] & & < \overline{b_{i,j}} > \arrow[ur, "\iota"] & & & < \overline{b_{n,n}} > \arrow[urr] & &
\end{tikzcd}$$

Note that in each case, $\mathcal{A}_k \in \mathcal{K}'$, so we can if necessary enlarge $\mathcal{A}_k$ and assume that $\mathcal{A}_k = \ell_\infty^{m_k}(\ell_1^{n_k})$ for some $m_k, n_k \in \mathbb{N}$.

Observe that for each tuple $\overline{a} \subseteq \mathcal{A}_k$, $\overline{b} \in K_{k,0}$, and each strictly approximate isometry $\psi : \overline{b} \times \overline{a} \to \mathbb{Q}$, we have some $m > k$ such that $\overline{a} = \overline{a_{i,j}}$, $\overline{b} = \overline{b_{i,j}}$ for some $i, j \leq m$ with $C_i^j \in C(\overline{a}, \overline{b})$ and embeddings $f :< \overline{a_{i,j}} > \to C_i^j$ and $\iota :< \overline{b_{i,j}} > \to C_i^j$ such that

$\psi_f^*\psi_\iota < \psi$. Additionally, there is an embedding $h : C_i^j \to \mathcal{A}_m$ with $\psi_h^*\psi_{\iota'} \leq \psi_f$. In particular, we have

$$\psi_{h\iota}|_{<\overline{b_{i,j}}>\times<\overline{a_{i,j}}>} = \psi_f^*\psi_\iota :$$

Indeed, given $x \in< \overline{a_{i,j}} >$ and $y \in< \overline{b_{i,j}} >$,

$$\begin{aligned}\psi_{h\iota}(y,x) &= \|h\iota(y) - x\| = \|h\iota(y) - \iota'(x)\| \\ &= \|h\iota(y) - hf(x)\| = \|\iota(y) - f(x)\| = \psi_f^*\psi_\iota.\end{aligned}$$

Thus $\psi_{h\iota} \leq \psi_f^*\psi_\iota < \psi$. Furthermore, $\psi_{h\iota}$ is r-total on $\overline{b_{i,j}}$ for all $r > 0$, since $h\iota$ is an embedding. Thus by [13, Lemma 3.16], $\overline{\bigcup_n \mathcal{A}_n}$ is a Fraïssé limit for $\overline{\mathcal{K}'}$-structures, where $\overline{\mathcal{K}'}$ is the Fraïssé completion of $\mathcal{K}'$. By Theorem 5.4.2, we have $\overline{\mathcal{K}'} = \mathcal{K}$. This implies that $\overline{\bigcup_n \mathcal{A}_n}$ is also a Fraïssé limit of $\mathcal{K}$, so by uniqueness, $\overline{\bigcup_n \mathcal{A}_n}$ is isometric to $\mathfrak{BL}$. $\square$

A lattice that is finitely branchable can be expressed as an inductive limit of finite dimensional lattices. We also have the following:

Theorem 5.4.4. *Any finitely branchable lattice can be generated by two elements.*

Proof. Let X be finitely branchable, and let $(x_\sigma)_{\sigma\in T}$ be a finitely branching tree densely spanning X. Recall that as a tree, $T \subseteq \bigcup_M \prod_n^M A_n$, where each A_n is a finite nonempty set. We will find two elements u and v such that $X =< u, v >$. Let $u = x_\emptyset$ and let $S_n = \{x_\sigma : |\sigma| = n\}$ as in the proof of Lemma 5.4.2. Consider now $X_1 = \text{span}(S_1)$, and let

$$v_1 = \sum_{|\sigma|=1} a_\sigma x_\sigma,$$

where $0 < a_\sigma$ and the a_σ's are mutually distinct. The mutual distinction enables each x_σ to be produced using lattice operations over u and v_1. For example, take $a_\rho = \max a_\sigma$, and pick c such that $c\alpha_\rho > 1$ but for all $\tau \neq \rho$, $ca_\tau < 1$. Recall that $u = x_\emptyset = \sum_{|\sigma|=1} x_\sigma$, so $(cv_1 - u) \vee 0 = (c\alpha_\rho - 1)x_\rho$. We then make the same argument, but for $u - x_\rho$ and $v_1 - a_\rho x_\rho$, thus generating each successive x_σ for all $\sigma \in S_1$.

Suppose that for all $k \leq n$, v_k has been selected and that for each k we have a finite sequence of functions $(\phi_k^i(x, y))_i$ generated by lattice operations $+, \wedge, r\cdot$ (where

r is real), with corresponding moduli of continuity $\Delta_k^i : \mathbb{R}^+ \to (0, 1]$. Suppose we also have:

- For each $k \leq n$, $v_k = \sum_{|\sigma|=k} a_\sigma x_\sigma$, with $a_\sigma > 0$ and mutually distinct.
- For each $k \leq n$, $< u, v_k >= \text{span}(S_k)$
- For each $k \leq n$, $(\phi_k^i(u, v_k))_i$ is a 2^{-k}-net in the unit ball of $\text{span}(S_k)$.
- For each $k < n$,

$$\|v_k - v_{k+1}\| < \frac{\min_{i,j \leq k}(\ (\Delta_j^i(2^{-k}))_i \ )}{2^k}$$

Note that

$$v_k = \sum_{|\sigma|=k} a_\sigma \sum_{m \in A_{k+1}} x_{\sigma^\frown m},$$

so for each $m \in A_{n+1}$ and $\sigma \in \prod_1^n A_k$, pick positive, mutually distinct $a_{\sigma^\frown m}$ such that

$$|a_\sigma - a_{\sigma^\frown m}| < \frac{\min_{i,j \leq n}(\ (\Delta_j^i(2^{-n}))_i \ )}{2^n |S_{n+1}|}.$$

Thus if $v_{n+1} := \sum_{|\sigma|=n+1} a_\sigma x_\sigma$, we have $\|v_n - v_{n+1}\| < \frac{\min_{i,j \leq n}(\ (\Delta_j^i(2^{-n}))_i \ )}{2^n}$. Now $< u, v_{n+1} >= \text{span}(S_{n+1})$. For each σ with $|\sigma| = n+1$, pick $0 < s < a_\sigma < r$ that for all $\tau \neq \sigma$ with $\tau \in \prod_1^{n+1} A_k$, either $\tau > r$ or $\tau < s$. Let $x = (v_{n+1} - su)_+$ and $y = (v_{n+1} - ru)_+$. Then for some large enough C, $(Cy - x)_+$ is a multiple of x_σ. Finally, let $(\phi_{n+1}^i(x, y))_i$ be a finite collection of functions generated by lattice operations such that $(\phi_{n+1}^i(u, v_{n+1}))_i$ is a 2^{-n-1}-net in the unit ball of $\text{span}(S_{n+1})$. Let $v = \lim v_n$.

We show that $< u, v >= X$. Observe that the set $\{\phi_n^i(u, v_n) : \ n \in \mathbb{N}\}$ is dense in X by the above properties. Let $\varepsilon > 0$, and pick n such that $2^{-n} < \varepsilon$. Then $\|v_n - v_{n+1}\| < \frac{\Delta_n^i(\varepsilon)}{2^{n+1}}$ and $\phi_n^i(u, v_n)$ is ε-dense in $\mathbf{B}(\text{span}(S_n))$. Furthermore, for all $m > n$, we have $\|v_n - v_m\| < \min_i \Delta_n^i(\varepsilon) \sum_{j=n+1}^m 2^{-j}$. Thus $\|v_n - v\| \leq \Delta_n^i(\varepsilon)$, so $\|\phi_n^i(u, v_n) - \phi_n^i(u, v)\| < \varepsilon$ for all ϕ_n^i. This implies that the set $\{\phi_n^i(u, v) | \ n \in \mathbb{N}\}$ is dense in X, so we are done. $\square$

Theorem 5.4.3 combined with Theorem 5.4.4 yields a surprising result:

Corollary 5.4.5. *The lattice $\mathfrak{BL}$ is finitely generated.*

Remark 5.4.6. Finite generation implies that $\mathfrak{BL}$ is not stably homogeneous in the sense defined by Lupini in [58]. That is, given finitely generated A and embeddings

$f : A \to \mathfrak{BL}$ and $g : A \to \mathfrak{BL}$, we cannot guarantee that for all $\varepsilon > 0$, there is some automorphism ϕ on $\mathfrak{BL}$ such that $\|\phi \circ f - g\| < \varepsilon$. Thus approximating over a finite number of elements rather than by norms is the best, in some sense, that can be done in terms of homogeneity.

Suppose otherwise. Since $\mathfrak{BL}$ can be generated by two elements x_1, x_2, we consider $e \in \mathbf{S}(\mathfrak{BL})_+$ and embedding $f :< x_1, x_2 >\to \mathfrak{BL}$ such that the image does not have full support and $f(e)$ is disjoint from e (we can do this, for example, by finding a copy of $\mathfrak{BL} \oplus_\infty \mathbb{R}$ that is in $\mathfrak{BL}$, and pick $e \in \mathbf{S}(\mathbb{R})_+$). If $\mathfrak{BL}$ were stably homogeneous, there would exist a lattice automorphism ϕ on $\mathfrak{BL}$ such that $\|f - \phi\| < \frac{1}{2}$. Then $\|f(\phi^{-1}(e)) - e\| = \|(f - \phi)(\phi^{-1}(e))\| < 1/2$, but $f(\phi^{-1}(e))$ is disjoint from e, which means $\|f(\phi^{-1}(e)) - e\| \geq 1$, a contradiction.

5.5 An alternate construction of $\mathfrak{BL}$ and some of its properties

The Fraïssé limit $\mathfrak{BL}$ is clearly of approximately universal disposition both for finite dimensional lattices and for finitely generated lattices. In this section, we show that separable lattices of approximately universal disposition for finitely generated lattices are isometric to $\mathfrak{BL}$. In addition, finitely branchable lattices of approximately universal disposition for finite dimensional lattices are isometric to $\mathfrak{BL}$. A bi-product of the latter is a simplified construction of $\mathfrak{BL}$ which allows us to explore some of its structural properties.

We first show that approximate universal disposition can be broadened to include extensions of ε-isometries:

Lemma 5.5.1. *Suppose X is of approximately universal disposition for finitely generated Banach lattices, let $< \overline{a} >= A \subseteq X$ and B be finitely generated, and let $f : A \to B$ be a $(1 + \varepsilon')$-embedding. Then for all $\varepsilon > \varepsilon'$ and for all $\delta > 0$, there exists a $(1 + \delta)$-embedding $g : B \to X$ such that $\|g \circ f(\overline{a}) - \overline{a}\| < \varepsilon$.*

Proof. By Theorem 5.3.13 there is a lattice Z and embeddings $h_1 : A \to Z$ and $h_2 : B \to Z$ such that $\|h_2 \circ f - h_1\| \leq \varepsilon'$. We can assume that Z is finitely generated as well, since we can embed into $\mathcal{U}$ if necessary. Decreasing δ as necessary, we can suppose that $(1 + \delta)\varepsilon' + \delta < \varepsilon$. Then there exists a $(1 + \delta)$-embedding $g' : Z \to X$

such that $\|g'h_1(\overline{a}) - \overline{a}\| < \delta$. Then

$$\|g'h_2f(\overline{a}) - \overline{a}\| \leq \|g'\|\|h_2(\overline{a}) - h_1(\overline{a})\| + \|g'h_1(\overline{a}) - \overline{a}\| < (1+\delta)\varepsilon' + \delta < \varepsilon.$$

Let $g = g' \circ h_2$, and we are done. $\square$

We can now show the following:

Theorem 5.5.2. *Any separable Banach lattice of approximately universal disposition for finitely generated lattices is isometric to* $\mathfrak{BL}$.

Proof. The proof follows that of Theorem 1.1 in [49]. Suppose X and Y are lattices of approximately universal disposition for finitely generated lattices. We will then construct a lattice isometry. Let (x_n), (y_n) be dense in X and Y, with $x_1 = 0$ and $y_1 \geq 0$. Given $\varepsilon > 0$, let $\varepsilon_n \downarrow 0$ be a decreasing sequence such that $\varepsilon_n < 2^{-n-1}$. Throughout, we let $\overline{x_n} = (x_1, ..., x_n)$, and let $X_n =< \overline{x_n} >$, with the same notation for $\overline{y_n}$ and Y_n. Finally, let $f_1 : X_1 \to Y_1$ be the trivial isometry.

We begin our construction: let $g_1 : Y_1 \to X$ be a $(1+\varepsilon_1)$-isometry. Note this isometry exists. Now take $\tilde{X_2} = BL(X_2 \cup g_1(Y_1))$. This lattice is also finitely generated by $\tilde{x_2} = \overline{x_2} \cup g_1(\overline{y_1})$, so we have the map $g_1 : Y_1 \to \tilde{X_2}$, and by Lemma 5.5.1, pick a $(1+\varepsilon_2)$-embedding $f_2 : \tilde{X_2} \to Y$ such that $d(f_2g_1(\overline{y_1}), \overline{y_1}) < \frac{1}{2^2}$. Now let $\tilde{Y_2} = BL(f_2(\tilde{X_2}) \cup Y_2)$. Use Lemma 5.5.1 again to generate a $(1+\varepsilon_3)$- embedding $g_2 : \tilde{Y_2} \to X$ such that $d(g_2f_2(\tilde{x_2}), \tilde{x_2}) < \frac{1}{2^3}$.

We can proceed inductively by constructing finitely generated subspaces $\tilde{X_n} = BL(X_n \cup g_{n-1}(\tilde{Y}_{n-1}))$ and $\tilde{Y_n} = BL(Y_n \cup f_n(\tilde{X_n}))$ with corresponding tuples $\tilde{x_n} = \overline{x_n} \cup g_{n-1}(\overline{y_{n-1}})$ and $\tilde{y_n} = \overline{y_n} \cup f_n(\overline{y_n})$, as well as $(1+\varepsilon_{2(n-1)})$-embeddings $f_n : \tilde{X_n} \to \tilde{Y_n}$ and $(1+\varepsilon_{2n-1})$-embeddings $g_n : \tilde{Y_n} \to \tilde{X}_{n+1}$ such that $d(\tilde{y_n}, f_{n+1}g_n(\tilde{y_n})) < \frac{1}{2^{2(n-1)}}$ and $d(\tilde{x_n}, g_nf_n(\tilde{x_n})) < \frac{1}{2^{2n-1}}$. Note that for each $k \leq n$, we have

$$\begin{aligned}
&\|f_{n+1}(x_k) - f_n(x_k)\| \\
=&\|f_{n+1}(x_k - g_nf_n(x_k) + g_nf_n(x_k)) - f_{n+1}(x_k)\| \\
\leq&\|f_{n+1}(x_k - g_nf_n(x_k))\| + \|(f_{n+1}g_n - Id)f_n(x_k)\| \\
\leq&\frac{2}{2^{2(n-1)}} + \frac{2}{2^{2(n-1)}} = \frac{1}{2^{2(n-2)}}
\end{aligned}$$

so the sequence $f_n(x_k)_{n\geq k}$ is Cauchy. The same is true for $g_n(y_k)_{n\geq k}$. Let $f = \lim f_n$, and let $g = \lim g_n$. These exist, since (x_k) and y_k are dense in X and Y. Furthermore,

f and g are inverses of each other, and they are each isometries. $\square$

We now construct a separable lattice of approximately universal disposition for finite dimensional lattices. The approach is a modification of that in [7, Section 5] for the Gurarij space.

Let $\mathfrak{J}$ be the collection of embeddings between finite dimensional lattices in $\mathcal{K}$. Since any such lattice isometrically embeds into $\mathcal{U}$, we can assume that $\mathfrak{J}$ is a set by limiting it to embeddings between finite dimensional sublattices of $\mathcal{U}$. Let $\mathfrak{J}_0$ be a countable dense subset of $\mathfrak{J}$ in the following sense: for all embeddings $f : A \to B$ with B finite dimensional and for all $\varepsilon > 0$, there exists $u : A' \to B' \in \mathfrak{J}_0$, and $(1+\varepsilon)$-isometries $\iota_A : A \to A'$ and $\iota_B : B \to B'$ such that $u \circ \iota_A = \iota_B \circ f$. In addition, for a separable lattice X, let $\mathfrak{L}(X)$ be the set of all maps $v : A' \to X$ which are C-embeddings for some $C \geq 1$ with $A' \in \mathrm{Dom}(\mathfrak{J}_0)$. Let $\mathfrak{L}_0$ be a countable subset of $\mathfrak{L}(X)$ which is dense in the following sense: for all $\varepsilon > \varepsilon' > 0$ and $(1+\varepsilon')$-embeddings $f : A' \to X$ with $A' \in \mathrm{Dom}(\mathfrak{J}_0)$, $\mathfrak{L}_0$ contains an $(1+\varepsilon)$-embedding $v : A' \to X$ such that $\|v - f\| < \varepsilon$.

Let $X_{0,0} = X$, and suppose now that $X_{n,k}$ has been constructed. Let $\mathfrak{L}_n$ be a countable subset of $\mathfrak{L}(X_{n,0})$ which is dense in the manner described for $\mathfrak{L}_0$ and $\mathfrak{L}(X)$. Finally, let $\Gamma_n = \{(u,v) \in \mathfrak{J}_0 \times \mathfrak{L}_n : \mathrm{dom}(u) = \mathrm{dom}(v)\}$, and let $((u_i^n, v_i^n))_i$ be an enumeration of Γ_n. We then construct $X_{n,k+1}$ by amalgamating as follows. Given $(u_k^n, v_k^n) \in \Gamma_n$ with v_k^n a C-embedding, we use part 2 of Corollary 5.3.12 to get an embedding $\iota : X_{n,k} \to X_{n,k+1}$ and C-embedding $w : \mathrm{cod}(u_k^n) \to X_{n,k+1}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{dom}(u_k^n) \arrow[r, "u_k^n"] \arrow[d, "v_k^n"] & \mathrm{cod}(u_k^n) \arrow[d, "w"] \\
X_{n,k} \arrow[r, "\iota"] & X_{n,k+1}
\end{tikzcd}$$

Finally, we let $X_{n+1,0} = \overline{\bigcup_{k\in\mathbb{N}} X_{n,k}}$, and then let $X_{\omega_0}(X) = \overline{\bigcup_{n\in N} X_{n,0}}$.

Theorem 5.5.3. *$X_{\omega_0}(X)$ is of approximately universal disposition for finite dimensional lattices.*

Proof. Let $f : A \to B$ and $g : A \to X_{\omega_0}(X)$ be embeddings, with A and B finite dimensional. Given $\varepsilon > 0$, by density of embeddings and spaces in $\mathfrak{J}_0$ and $\mathfrak{L}_n$, pick an embedding $u : A' \to B'$ with $u \in \mathfrak{J}_0$ such that there are $(1+\varepsilon/2)$-isometries $i_1 : A \to A'$ and $i_2 : B \to B'$ with $u \circ i_1 = i_2 \circ f$. Since $g \circ i_1^{-1}$ is also a $(1+\varepsilon/2)$-embedding, there exists n and a $(1+\varepsilon)$-embedding $v : A' \to X_{n,0}$ such that $v \in \mathfrak{L}_n$ and $\|v - g \circ i_1^{-1}\| < \varepsilon$. Then $\|v \circ i_1 - g\| < (1+\varepsilon)\varepsilon$. Using the construction above, we thus have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_1"] & B \arrow[d, "i_2"] \\
A' \arrow[r, "u"] \arrow[d, "v"] & B' \arrow[d, "\iota_{B'}"] \\
X_{n,0} \arrow[r, "\iota"] & X_{n+1,0}
\end{tikzcd}$$

In the diagram, i_1, i_2, v and $\iota_{B'}$ are each $(1+\varepsilon)$-embeddings. We now let $h : B \to X_{\omega_0}(X) = \iota_{B'} \circ i_2$. This is clearly a $(1+\varepsilon)^2$-embedding. Finally, for all $x \in \mathbf{B}(A)$, we have

$$\begin{aligned}
\|g(x) - hf(x)\| &= \|g(x) - \iota_{B'} i_2 f(x)\| \\
&= \|g(x) - vi_1(x)\| < (1+\varepsilon)\varepsilon.
\end{aligned}$$

Thus $X_{\omega_0}(X)$ is of approximately universal disposition for finite dimensional lattices.

$\square$

One can use a similar argument to construct (non-separable) lattices of universal disposition by amalgamating over all combinations of embeddings of separable spaces ω_1 times rather than selecting a countable subset each step (see [8, Theorem 5.3]).

We can adapt our construction with additional conditions as a way to discern the structure of $\mathfrak{BL}$. For instance, we can start with $X \in \mathcal{K}'$ in particular, and then inductively construct increasing lattices $X = X_0 \subseteq X_1 \subseteq ... \subseteq X_n \subseteq X_{n+1} \subseteq ....$ with each $X_n \in \mathcal{K}'$. First, we ensure that $\mathfrak{J}_0$ and each $\mathfrak{L}_n$ consist only of maps between lattices in $\mathcal{K}'$. This is possible assuming X_0 is in $\mathcal{K}'$ and by Theorem 2.5.4. Then given $X_n \in \mathcal{K}'$, we can construct $X_{n+1} \in \mathcal{K}'$ by applying parts 2 and 3 of Corollary 5.3.12 over the n^{th} pair $(u_n^k, g_n^k) \in \Gamma_k$ for each $k < n$, followed by the first n pairs in Γ_n.

$$\begin{tikzcd}
\mathrm{dom}(u_n^1) \arrow[r, "u_n^1"] \arrow[d, "v_n^1"] & \mathrm{cod}(u_n^1) \arrow[dr] & \mathrm{dom}(u_n^2) \arrow[r, "u_n^2"] \arrow[d, "v_n^2"] & \mathrm{cod}(u_n^2) \arrow[dr] & \dots & \mathrm{dom}(u_n^n) \arrow[r, "u_n^n"] \arrow[d, "v_n^n"] & \mathrm{cod}(u_n^n) \arrow[d] \\
X_n \arrow[rr] & & * \arrow[rr] & & * \arrow[r, "\cdots"] & * \arrow[r] & X_{n+1}
\end{tikzcd}$$

Thus for any pair $(u_n^k, v_n^k) \in \Gamma_k$ with v_n^k a C-embedding for some C, there exists some $m > k$ (here $m = \max(k+1, n+1)$) and a C-embedding $\iota : \mathrm{cod}(u_n^k) \to X_m$

such that $v_n^k = \iota \circ u_n^k$. The lattice $\overline{\bigcup_n X_n}$ is then a limit of finite dimensional lattices and is thus finitely branchable, and a small variation of the argument in Theorem 5.5.3 can be used to show that it is also of approximately universal disposition for finite dimensional lattices. It turns out, however, that we have derived an alternate, simplified construction of $\mathfrak{BL}$:

Theorem 5.5.4. *Any two finitely branchable lattices of approximately universal disposition for finite dimensional lattices are isometric. In particular, they are isometric to $\mathfrak{BL}$ and are thus of approximately universal disposition for finitely generated lattices.*

Proof. Suppose X and Y are two finitely branchable separable lattices of approximately universal disposition for finite dimensional lattices. As in the proof of Theorem 5.5.2, we simply construct an isometry $f : X \to Y$ with its inverse $g : Y \to X$.

Let (X_n) and (Y_n) be sequences of finite dimensional lattices generated by corresponding spanning trees, (here we let $X_n = \operatorname{span}(\{x_\sigma : |\sigma| = n\})$ such that $X = \overline{\bigcup X_n}$ and $Y = \overline{\bigcup Y_n}$. Let $\varepsilon_n \downarrow 0$ be a sequence such that $\prod(1+\varepsilon_n) < 1+\varepsilon$. We then proceed just like in the proof of Theorem 5.5.2, but with a modification. Let $f_1 : X_0 \to Y_0$ be an isometry (this is possible because X_0 and Y_0 are simply 1-dimensional lattices spanned by $x_\emptyset$ and $y_\emptyset$, respectively). Now, let $g_1 : Y_1 \to X$ be a $(1+\varepsilon_1)$-embedding such that $\|x_\emptyset - g_1 f_1(x_\emptyset)\| \leq \frac{1}{2}$, By density of $\bigcup X_n$, and since Y_1 is finite dimensional, we can in fact ensure that g_1 maps into some X_{k_2} for some $k_2 \in \mathbb{N}$.

Rather than generating lattices $\tilde{X}_n$ and $\tilde{Y}_n$, using Lemmas 5.5.1 and 5.3.6, we can pick $(1+\varepsilon_{2(n-1)})$-embeddings $f_n : X_{k_n} \to Y_{k'_n}$ and $(1+\varepsilon_{2n-1})$-embeddings $g_n : Y_{k'_n} \to X_{k_{n+1}}$ such that $\|g_n f_n - Id_X\| < \frac{1}{2^{2n-1}}$, and similarly, $\|f_n g_{n-1} - Id_Y\| \leq \frac{1}{2^{2(n-1)}}$. Then

$$
\begin{aligned}
\|f_{n+1} - f_n\| =& \|f_{n+1} - f_{n+1}g_n f_n + f_{n+1}g_n f_n - f_n\| \\
\leq& \|f_{n+1}\|\|Id_X - g_n f_n\| + \|f_{n+1}g_n - Id_Y\|\|f_n\| \\
\leq& \frac{2}{2^{2(n-1)}} + \frac{2}{2^{2(n-1)}} = \frac{1}{2^{2(n-2)}}
\end{aligned}
$$

A similar argument can be made for the g_n's. Thus for all n and for all $x \in X_{k_n}$ the sequence $(f_m(x))_{m>n}$ is Cauchy. The same is true for any $y \in Y_{k'_n}$ and sequence $(g_m(y))_{m>n}$. Let $f = \lim f_n$ and $g = \lim g_n$, and we are done. □

The assumption of finite branchability is essential in the above proof. It is currently unknown, however, if there are lattices which are not finitely branchable but are of approximately universal disposition for finite dimensional lattices.

Since $\mathfrak{BL}$ is finitely branchable, it contains many non-trivial projection bands. Recall Section 1.2 that a ideal sublattice $B \subseteq X$ is a band if for all $x \in X$ and sets $A \subseteq B$, if $x = \sup A$, then $x \in B$. Given a set $A \subseteq X$, we let $A^{\perp} = \{x \in X : x \perp a \text{ for all } a \in A\}$. $A^{\perp}$ is itself a band, and if B is a band, then $B^{\perp\perp} = B$ (see [1, Theorem 1.28]). A band B is a **projection band** if $X = B \oplus B^{\perp}$; that is, every $x \in X$ can be uniquely written as $x_1 + x_2$ with $x_1 \in B$ and $x_2 \in B^{\perp}$. Note that if B is a projection band, it induces a lattice projection $P : X \to B$, that is, a contractive lattice homomorphism onto B with $P^2 = P$ and in particular, $P|_B = Id|_B$. Let $(x_\sigma)_{\sigma \in T}$ be a linearly dense spanning tree in $\mathfrak{BL}$. Then it is clear that $\mathfrak{BL} = \oplus_{|\sigma|=n}\overline{\text{span}}(\{x_\tau : \tau \supseteq \sigma\})$, and that each sublattice $\overline{\text{span}}(\{x_\tau : \tau \supseteq \sigma\})$ is a projection band. We thus have the following:

Theorem 5.5.5. *Every non-trivial projection band in $\mathcal{B} \subseteq \mathfrak{BL}$ is itself isometric to $\mathfrak{BL}$.*

Proof. First note that $\mathcal{B}$ itself is finitely branchable. Given a finitely branching tree $(x_\sigma)_{\sigma \in T} \subseteq \mathfrak{BL}$ and lattice projection $P : \mathfrak{BL} \to \mathcal{B}$, we get $(P(x_\sigma))_{\sigma \in T}$ as a spanning tree for $\mathcal{B}$.

We now show that $\mathcal{B}$ is of approximately universal disposition for finite dimensional lattices. By Theorem 5.5.4, this implies that $\mathcal{B}$ must in fact be isometric to $\mathfrak{BL}$.

Let $\mathfrak{J}_0$, Γ_n, $\mathfrak{L}_n$, and X_n denote the sets and lattices used in the construction preceding Theorem 5.5.4. Let $A \subseteq \mathcal{B}$, and $f : A \to B$ be an isomeric embedding between finite dimensional A and B. Given $\varepsilon > 0$, the goal is to construct a $(1+\varepsilon)$-embedding $g : B \to \mathcal{B}$ such that $\|g \circ f - Id|_A\| < \varepsilon$.

We begin with the case where $f(A)$ fully supports B. Let $(a_i)_i$ and $(b_i)_i$ be finite sequences enumerating the atoms of A and B, and let N be such that $Nf\big(\sum_i a_i\big) \geq \sum_i b_i$. Finally, let $x_{\mathcal{B}}$ be a weak unit in $\mathcal{B}$ such that $\sum_i a_i \leq x_{\mathcal{B}}$. Then given $\delta > 0$, by Lemma 5.3.6 and density of $\cup X_n$ in $\mathfrak{BL}$, there exist $m \in \mathbb{N}$, a $(1+\delta)$-embedding $v : A' \to X_m$ with $v \in \mathfrak{L}_m$, $u : A' \to B'$ with $u \in \mathfrak{J}_0$, $j_1 : A \to A'$, $j_2 : B \to B'$, and $x'_B \in X_m$ such that

1. For all i, $\|Na_i - Nvj_1(a_i)\| < \frac{\delta}{\dim A}$
2. j_1 and j_2 are $(1+\delta)$- isometries with $u \circ j_1 = j_2 \circ f$
3. $\|Nx_{\mathcal{B}} - Nx'_B\| < \delta$ and $x'_B \geq vj_1\big(\sum_i a_i\big)$.

If necessary, we can replace $x_{\mathcal{B}}$ with $x_{\mathcal{B}} \vee P(x'_B)$. All prior conditions will still be fulfilled, so we can assume $x_{\mathcal{B}} \geq P(x'_B)$. Now $(u, v) \in \Gamma_m$, so there exists some $n > m$ such that B' also $(1+\delta)$-embeds into X_n:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow{\scriptstyle j_1} & & \Big\downarrow{\scriptstyle j_2} \\ A' & \xrightarrow{u} & B' \\ \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle \iota_{B'}} \\ X_m & \xrightarrow{\iota} & X_n \end{array}$$

Furthermore,

$$(*)\ Nx'_B \geq N\sum_i vj_1(a_i) = N\sum_i \iota vj_1(a_i) = N\sum_i \iota_{B'}uj_1(a_i)$$
$$= \sum_i \iota_{B'}\iota_2\big(N\sum_i f(a_i)\big) \geq \iota_{B'}\iota_2\big(\sum_i b_i\big).$$

.

Let $g : B \to \mathcal{B}$ be defined by $g = P \circ \iota_{B'} \circ j_2$. By $(*)$, we have that $Nx_{\mathcal{B}} \geq NP(x'_B) \geq g\big(\sum_i b_i\big)$. Furthermore, since P is a band projection, it follows that for all $\sum_i c_i\iota_{B'}j_2(b_i)$ with $0 \leq c_i \leq 1$,

$$Nx_{\mathcal{B}} \wedge \big(\iota_{B'}j_2\sum_i c_ib_i\big) = Nx_{\mathcal{B}} \wedge P\iota_{B'}j_2\big(\sum_i c_ib_i\big) = g\big(\sum_i c_ib_i\big).$$

Therefore

$$\left\|\sum_i c_i\iota_{B'}j_2(b_i) - \sum_i c_ig(b_i)\right\| =$$
$$\left\|Nx'_B \wedge \sum_i c_i\iota_{B'}j_2(b_i) - Nx_{\mathcal{B}} \wedge \sum_i c_i\iota_{B'}j_2(b_i)\right\| < \delta,$$

so $\|\iota_{B'} \circ j_2 - g\| < \delta$, and since $\iota_{B'} \circ j_2$ is a $(1+\delta)^2$-embedding, g is a $\frac{(1+\delta)^3}{1-\delta}$-embedding. Finally, since the diagram commutes, we have $gf(a_i) = P\iota_{B'}j_2f(a_i) = Pvj_1(a_i)$. Since $a_i \in \mathcal{B}$, $P(a_i) = a_i$, so for all $\sum_i c_ia_i \in \mathbf{S}(A)$, by condition 1 we have

$$\Big\| gf\big(\sum_i c_i a_i\big) - \sum_i c_i a_i \Big\| =$$
$$\Big\| \sum_i c_i \big(P v j_1(a_i) - P(a_i)\big) \Big\| \leq \sum_i c_i \| v j_1(a_i) - a_i \| < \delta.$$

Now δ can be arbitrarily small, so assume that $\frac{(1+\delta)^3}{1-\delta} < 1+\varepsilon$. Since $g = P \circ \iota_{B'} \circ j_2$, it sends elements of B into $\mathcal{B}$ thanks to composition by P. Thus g satisfies the requirements.

Suppose now that B is not fully supported by A. For all $\delta > 0$, we can perturb f with a $(1+\delta)$-embedding $f' : A \to B$ such that $f'(A)$ fully supports B and $\|f - f'\| < \delta$. By Lemma 5.3.11, let B' be a copy of B with a $(1+\delta)$-equivalent renorming so that $f' : A \to B'$ is an embedding.

Now use the result above to get a $(1+\delta)$ embedding $g : B' \to \mathcal{B}$ such that $\|Id|_A - g \circ f'\| < \delta$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f'"] \arrow[dd, "Id|_A"'] & B \arrow[d, "Id_B"] \\
 & B' \arrow[dl, "g"] \\
\mathcal{B} &
\end{tikzcd}$$

Now with the original norm on B, g is a $(1+\delta)^2$-embedding, and under the norm on B', $\|f - f'\|_{B'} < (1+\delta)\delta$. Then

$$\|g \circ f - Id\| \leq \|g \circ (f - f')\| + \|g \circ f' - Id\| \leq \|g\| \|f - f'\|_{B'} + \delta < (1+\delta)^2 \delta + \delta.$$

We can then let δ be arbitrarily small. Thus $\mathcal{B}$ is of approximately universal disposition for finite dimensional lattices. $\square$

5.6 Constructing a Pelczynski lattice of almost universal disposition

Throughout this section we will say that a lattice homomorphism embedding $f : X \to Y$ is an **ideal homomorphism** if $f(X)$ is a lattice ideal in Y. Observe that if X and Y are atomic, then f is an ideal homomorphism if f maps the atoms of X to the atoms of Y. Along the same vein, we also refer to ideal (C)-embeddings, though clearly any ideal C-isometry is simply a C-isometry. In this section, we will work specifically with atomic order continuous lattices and assume that ideal embeddings map unit atoms to unit atoms.

Recall from Theorem 3.5.2 in Section 3.5 the existence of an separable order continuous atomic Pelczynski lattice which isomorphically contains any order continuous atomic sublattice as an ideal with arbitrarily small distortion. This lattice is isomorphically unique. Here, we use a different technique to construct a Pelczynski lattice $\mathfrak{U}$ which in addition is of almost universal disposition in the following sense: For all finite dimensional lattices $A \subseteq B$ with A ideal in B, for all $C \geq 1$ and $\varepsilon > 0$, and for all ideal embeddings $f : A \to B$, there exists an ideal embedding $g : B \to \mathfrak{U}$ extending f. We will also show that the lattice with these properties is C-isometrically unique for all $C > 1$, but is not isometrically unique.

Similar results have been shown with respect to various classes of Banach spaces and linear "basis-preserving" maps. For instance, [9] and [10] construct isometrically unique Banach spaces which are isomorphically universal for the classes of K-unconditional and K-suppression spaces V_u and V_s that have homogeneity properties like the one above. Specifically, for any "basis-preserving" linear isometry $g : A \subseteq B$, with A and B finite dimensional K-unconditional (K-suppression unconditional) rational spaces, and basis-preserving linear isometry $f : A \to V_u$ (V_s), there is a basis preserving linear $h : B \to V_u$ (V_s) such that $f = h \circ g$. The use of rational finite dimensional spaces (that is, Banach spaces whose unit balls are polyhedra whose vertices are rationally valued coordinates) plays a role in the isometric uniqueness of such spaces. The homogeneity property for rational spaces can be extended to almost universal disposition for any finite dimensional space in the appropriate class, since the unit ball of any Banach space can be approximated by that of a rational space with arbitrarily small distortion. The approach in both papers involves the use of countable Fraïssé sequences described in [47]. A similar approach can be used here, but we will employ the method used in Section 5.6.

We first observe that a simplified amalgamation property holds for finite dimensional lattices and ideal embeddings:

Lemma 5.6.1. *Let $f_1 : E \to F_1$ and $f_2 : E \to F_2$ be ideal C-embeddings with $C \geq 1$. Then there exist H and C-embeddings $g_i : F_i \to H$ with $g_1 f_1 = g_2 f_2$ such that g_i sends unit atoms in F_i to unit atoms in H.*

Proof. Let $H = f_1(E)^{\perp} \oplus F_2$. For $x \in f_i(E)^{\perp}$ and $f_i(y) \in f_i(E)$, let $g_i(x + f_i(y)) = x + f_2(y)$. Clearly $g_1 f_1 = g_2 f_2$. By Theorem 5.3.11, for each i, let $\tilde{\mathbf{B}}(F_i)$ be the unit ball obtained by C-equivalent norms $|||\cdot|||_i$ on F_i such that $f_i : E \to (F_i, |||\cdot|||_i)$ are ideal embeddings. Then let the norm of H be that induced by having

$$\mathbf{B}(H) = \mathrm{SCH}(g_1(\tilde{\mathbf{B}}(F_1)) \cup g_2(\tilde{\mathbf{B}}(F_2))).$$

Since $g_1(\tilde{\mathbf{B}}(F_1)) \cup g_2(\tilde{\mathbf{B}}(F_2))$ is compact, $\mathbf{B}(H)$ is closed. We now show that each g_i is a C-ideal embedding, by proving that each $g_i : (F_i, |||\cdot|||_i) \to H$ is an ideal embedding. Clearly it is an ideal homomorphism. Suppose $|||x + f_1(y)|||_i = 1$. Then $g_1(x+f_1(y)) = x+f_2(y) \in \mathbf{B}(H)$, so $\|g_1(x+f_1(y))\|_H \leq 1$. Now, Let $|||x + f_1(y)|||_1 = 1$, and let $M \geq 1$ be such that

$$M(x + f_2(y)) \leq t(x_1 + f_2(y_1)) + (1 - t)f_2(y_2),$$

with $f_2(y_2) \in \tilde{\mathbf{B}}(F_2)$ and $x \in F_1$. Then $|||f_1(y_2)|||_1 = \|y_2\|_E = |||f_2(y_2)|||_2 \leq 1$, so $|||tx_1 + f_2(ty_1 + (1-t)y_2)|||_1 \leq 1$. This implies that $M \leq 1$ as well. So $\|g_1(x + f_1(y))\|_H = \|x + f_2(y)\|_H \geq 1$. Thus $g_1 : (F_1, |||\cdot|||) \to H$ is an ideal embedding, implying that $g_1 : F_1 \to H$ is a C-ideal embedding. One can use a similar argument to prove the same for g_2.

□

We now construct a Pelzcynski lattice of almost universal disposition as follows: Start with a finite dimensional lattice $X = X_0$. We let $\mathfrak{J}$ be countably dense in the set of ideal embeddings betweem finite dimensional lattices in the following sense: for all ideal embeddings $f : A \to B$ and $\varepsilon > 0$, there exists an ideal embedding $u : A' \to B' \in \mathcal{J}$ and $(1+\varepsilon)$-isometries $\iota_A : A \to A'$ and $\iota_B : B \to B'$. such that $u \circ \iota_A = \iota_B \circ f$. Similarly, we let $\mathfrak{L}(X)$ be the collection of ideal embeddings from $A' \in \mathrm{dom}(\mathfrak{J})$ into X. Observe that if X is finite dimensional, then $\mathfrak{L}(X)$ is necessarily countable. Finally, we let $\mathfrak{d}(X) = \{(u, v) \in \mathfrak{J} \times \mathfrak{L}(X) : \mathrm{dom}(u) = \mathrm{dom}(v)\}$.

Suppose we have constructed X_n. Construct X_{n+1} by taking the $(n+1)^{th}$ pair $(u_n^k, v_n^k) \in \Gamma(X_k)$ for all $k < n$), as well as the first $n+1$ pairs $(u_i^n, v_i^n) \in \Gamma(X_n)$, and use Lemma 5.6.1 and Corollary 5.3.12 to amalgamate repeatedly with ideal embeddings. The same diagram 5.5 illustrates how this works. Finally, let $\mathfrak{U} = \overline{\cup_n X_n}$. Observe here that each X_n is ideal in $\mathfrak{U}$.

Theorem 5.6.2. *$\mathfrak{U}$ is a lattice of almost universal disposition for finite dimensional lattices with ideal embeddings as maps.*

Proof. Suppose $f : (a_1, ..., a_n) \mapsto (v_{k_1}, ..., v_{k_n})$ and $g : (a_1, ..., a_n) \mapsto (b_1, ..., b_n)$ induce embeddings from A into $A' \subseteq \mathfrak{U}$ and A to B, respectively, and let $\varepsilon > 0$. For some N, we have $A \subseteq X_N$, so find $(u, v) \in \Gamma(X_n)$ such that $A' = (dom)(u)$, B is $(1+\varepsilon)$-isometric to $B' = \mathrm{cod}(u)$, and $g \circ \iota_A$ so that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[dr, "j_1"] \arrow[ddr, "f"'] & B \arrow[dr, "j_2"] & \\
& A' \arrow[r, "u"] \arrow[d, "v"] & B' \\
& X_n &
\end{tikzcd}$$

Then there exists $M > N$ such that B' ideally embeds into $\mathfrak{U}$, so we have the following:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[dr, "j_1"] \arrow[ddr, "f"'] & B \arrow[dr, "j_2"] & \\
& A' \arrow[r, "u"] \arrow[d, "v"] & B' \arrow[d, "\iota_{B'}"] \\
& X_N \arrow[r, "\iota"] & X_m
\end{tikzcd}$$

Let $h : \iota_{B'} \circ j_2$. h is a $(1+\varepsilon)$-ideal embedding and $f = h \circ g$, so we are done.

□

Corollary 5.6.3. *For all ideal C-embeddings $f : A \to \mathfrak{U}$ with $A \subseteq B$ finite dimensional and for all $\varepsilon > 0$, there exists an ideal $(C+\varepsilon)$-embedding $g : B \to \mathfrak{U}$ extending f.*

Proof. Let $A' = f(A)$, and let B' be a C-equivalent renorming of B such that A' embeds into B'.

$$
\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "f"] \arrow[ddr, "f"'] & B \arrow[dr, "\iota_B"] & \\
& A' \arrow[r, "\iota"] \arrow[d, "\iota_{A'}"] & B' \arrow[dl, "h"] \\
& \mathfrak{U} &
\end{tikzcd}
$$

Find an ideal $(1+\varepsilon/C)$-embedding $h : B' \to \mathfrak{U}$ extending $\iota_{A'}$. Then $g = h \circ \iota_{A'}$ is a $(C+\varepsilon)$-embedding extending f.

$\square$

The above corollary allows us now to prove the following:

Theorem 5.6.4. *$\mathfrak{U}$ is a Pelczynski lattice: i.e., for all separable order continuous atomic X generated by atoms $(e_1, ..., e_2)$ and for all $C > 1$, there is a subsequence $(\upsilon_{k_n})_n \subseteq (\upsilon_k)_k$ such that the map induced by $e_n \to \upsilon_{k_n}$ is a C-embedding.*

Proof. start with $e_1 \mapsto \upsilon_1$. Let $1 < C_1 < C_2 < .... < C$ be an increasing sequence bounded by C, and let X_n be the lattice generated by $(e_1, ..., e_n)$. For each n, suppose we have a C_n-embedding induced by $f_n : e_m \mapsto \upsilon_{k_m}$ for $m \leq n$. Then by Corollary 5.6.3, f_n can be extended to a ideal C_{n+1}-embedding extending f_n induced by mapping e_{n+1} to some $\upsilon_{k_{n+1}}$. Then let $f = \overline{\cup_n f_n}$. Now $f|_{X_n} = f|_n$, and span$((e_n)_n)$ is dense in X, so f is an ideal C-embeddding from X into $\mathfrak{U}$. $\square$

Finally, we present a result and some final commentary on the uniqueness of $\mathfrak{U}$:

Theorem 5.6.5. *Suppose $\mathfrak{W}$ is a separable atomic order continuous lattice generated by atoms $(w_n)_n$ with the following two properties:*

1. *For all $C > 1$ and for all separable atomic order continuous atoms X generated by $(e_n)_n$, there exists a subsequence $(w_{k_n})_n$ such that the map $e_n \mapsto w_{k_n}$ induces a C-embedding from X into $\mathfrak{W}$.*

2. *For all finite dimensional $A \subseteq B$, for all $C > 1$ and $\varepsilon > 0$, and for all C-embeddings $f : A \to \mathfrak{W}$ induced by atoms $a_n \mapsto w_{k_n}$ there exists an ideal $(C+\varepsilon)$-embedding $g : B \to \mathfrak{W}$ extending f.*

Then for all $C > 1$, $\mathfrak{W}$ is C-isometric to $\mathfrak{U}$.

Proof. Apply a back and forth argument over the atoms in $\mathfrak{U}$ and $\mathfrak{W}$. $\square$

Remark 5.6.6. The theorem cannot be strengthened to show full isometric uniqueness. There are only countably many ideals generated by finite dimensional lattices, while there are uncountably many isometrically distinct ℓ_p^n alone for $n > 1$.

Remark 5.6.7. Observe that the method of constructing $\mathcal{V}$ in Section 3.5 does not on its own guarantee the properties of universal disposition found in $\mathfrak{W}$. It turns out that Pelczynkski lattices, while isomorphically unique, are not necessarily C-isometrically unique for all $C > 1$. Indeed, consider the lattice $W = \mathbb{R} \oplus_1 \mathfrak{U}$, and let e_0 be the atom generating the left $\mathbb{R}$ in W. Now $< e_0, e_k >$ is isometric to ℓ_1^2 for all $k \in \mathbb{N}$. Thus if $T : \mathfrak{U} \to W$ with $T(e_k) = e_0$, then for all $C > 1$, there is some $e_{k'}$ with $< e_k, e_{k'} >$ C-isometric to ℓ_∞^2, but $< T(e_k), T(e_{k'}) >$ is isometric to ℓ_1^2. This implies that T will have distortion at least $C = 2$.

5.7 Questions and further research

Question 5.7.1 (Universal disposition for finitely generated vs. finite dimensional lattices)**.** One can construct Banach lattices of (full) universal disposition for the classes of finite dimensional and finitely generated Banach lattices. However, are these lattices actually isometric? Does there exist a separable lattice of approximately universal disposition for finite dimensional lattices which itself is not finitely branchable? A possible approach is to consider some modified version of $X_{\omega_0}(C([0, 1]))$. Maybe we can adapt the amalgamation in some way so that we don't "break" up the interval in the process?

Question 5.7.2 (How "fully" homogeneous is $\mathfrak{BL}$?)**.** The Gurarij lattice is only approximately homogeneous, given the existence of weak units in separable lattices. Can we get full homogeneity or almost universal disposition when we restrict to elements of full support? Similarly, can we get full homogeneity when both lattices are not fully supported?

It is known, for instance, that the atomless space $L_p(0, 1)$ is itself a metric Fraïssé limit for the class of finitely generated L_p spaces (see [34] for more treatment on this). It turns out that there are two orbit equivalence classes in the action induced by $Iso(L_p(0, 1))$ on $\mathbf{S}(L_p(0, 1))_+$: one class for fully supported elements and one for elements without full support.

Question 5.7.3 (Positioning of a sublattice into a lattice)**.** One can also explore the "opposite" end of homogeneity by examining the positions, up to isometry, of a sublattice in an ambient lattice. Suppose X and Y are lattices with Y embedding into X. Let $Emb(Y, X)$ be the set of lattice embeddings from Y to X with infinite co-dimension, and equip it with the strong operator topology. Then $Emb(Y, X)$ is a standard Borel space. For $f, g \in Emb(Y, X)$, let $f \sim g$ iff there exists an lattice automorphism $h : X \to X$ with $h \circ f = g$. Essentially, we are looking at how the positions of Y relate to each other within X. This is clearly an equivalence relation which leads to the following question: What is the complexity of the equivalence relation $\sim$ for select $Y \subseteq X$?

Anisca, Ferenczi, and Moreno explored this question for Banach spaces in [5]. The investigation on positions of subspaces is in part motivated by a classical result that shows that any isometry $f : Y \to Y'$ where Y and Y' are infinite codimensional subspaces of c_0 extends to an automorphism of c_0. The same is true for ℓ_2. These spaces are known as automorphic spaces. However, it is not known whether there are any other automorphic spaces, and even simple choices of X and Y can yield complex positions. In particular, [5] gives certain conditions on X where there is some Y such that the equivalence of positions of Y in X Borel reduces E_0. In addition, E_1 is Borel reducible to $\sim$ in $Emb(\ell_p, \ell_p)$, meaning that $\sim$ cannot be reduced to a relation induced by a group action of a Polish group.

We can modify this equivalence with a looser one by allowing for almost commutativity. That is, let $f \sim_A g$ iff for all $\varepsilon > 0$, there exists some automorphism $h : Y \to X$ such that $\|h \circ f - g\| < \varepsilon$. Then what is the complexity of the equivalence relation $\sim_A$ for select $Y \subseteq X$? The complexity of $\sim_A$ over various Y is an indication, then, of the amount of approximate homogeneity in X. For example, if $X = \mathfrak{BL}$ or L_p and Y is finite dimensional, then $\sim_A$ has only one equivalence class.

Question 5.7.4 (Fraïssé subclasses)**.** Are there other interesting classes of finitely generated lattices that are Fraïssé? It is already known that the class of L_p lattices are Fraïssé, but their limit is itself finitely generated We can either stay in the separable realm (say, just consider p-convex or q-concave lattices), or branch out to non-separable classes.

Question 5.7.5 (Universality and automorphism groups of $\mathfrak{BL}$)**.** Under certain conditions, both automorphism groups of ultra-homogeneous and approximately ultra-homogeneous spaces have certain properties. For instance, both the isometry group of the Urysohn space $\mathbb{U}$ and the linear isometry group over the Gurarij space $\mathfrak{G}$ are isomorphically universal for Polish groups, that is, every Polish group is homeomorphic to a (closed) subgroups of $Iso(\mathbb{U})$ and $Iso(\mathfrak{G})$ (see [71] and [12], respectively). One can also ask the same for $\mathfrak{BL}$: Is $Iso(\mathfrak{BL})$ a universal Polish group?

For both $\mathbb{U}$ and $\mathfrak{G}$, the associated proofs used Katetov functions, or some variation thereof, to show how increasing sequences of spaces also induced increasing sequences of Isometry groups. However, universality of the automorphism group of a Fraïssé structure for appropriate class is not automatically a given. For the classical Fraïssé limits, see [63] for a characterization of Fraïssé limits with universal groups, as well as [48] for some counter-examples.

Chapter 6

Eliminating non-trivial isometries in AM-spaces

6.1 Introduction

This chapter is based on the material in [65], a recently submitted work co-authored with Oikhberg treating on the renorming of AM-spaces. The question of renormings has been extensively studied in the Banach space literature. The goal is to equip a prescribed Banach space with an equivalent norm in a way that alters its isometric properties in a certain desirable way (the isomorphic properties meanwhile remain the same). Many results of this type appear in [24]; for more modern treatment see [32] or [31].

We are interested in producing a renorming with a prescribed group of isometries (throughout this chapter, all isometries are assumed to be linear and surjective unless specified otherwise). One of the first results appeared in [11]cha; there, it was shown that any separable real Banach space can equipped with an equivalent norm for which there are only two isometries – the identity and its opposite. The separability assumption was later removed in [44]. More recent papers [33], [36], [37] deal with renorming a separable Banach space in a way that produces a prescribed group of isometries.

We consider lattice renormings of separable AM-spaces. Recall that an AM-space is a Banach lattice in which the equality $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for any positive x and y; a lattice norm with this property is called an AM-norm. We also restrict oursleves to lattice isometries – that is, surjective (linear) isometries which preserve lattice operations. Our main result is:

Theorem 6.1.1. *Suppose $(X, \|\cdot\|)$ is a separable AM-space, and $c > 1$. Then X can be equipped with an equivalent lattice norm $|\!|\!|\cdot|\!|\!|$ so that $\|\cdot\| \leq |\!|\!|\cdot|\!|\!| \leq c\|\cdot\|$, and the identity map is the only lattice isometry on $(X.|\!|\!|\cdot|\!|\!|)$. If X has no more than one atom, then $|\!|\!|\cdot|\!|\!|$ can be chosen to be an AM-norm.*

The restriction on the number of atoms is essential; see Remark 6.6.8.

The proof of Theorem 6.1.1 proceeds in two steps. In Section 6.2, we introduce a new class of AM-spaces, which we call "Benyamini spaces," for their original discoverer [15]. We establish that any separable AM-space can be transformed into a Benyamini space with arbitrarily small distortion; this result may be interesting in its own right. In Section 6.6 we renorm a Benyamini space, eliminating all non-trivial isometries (the new norm may cease to be an AM-norm if more than one atom is present). The proof of Theorem 6.1.1 then follows by combining Proposition 6.2.3 and Theorem 6.6.1.

Throughout this chapter, we use the standard functional analysis facts and notation. For more detail, the reader is referred to e.g. [55] and [61]. All spaces are assumed to be separable, and the field of scalars is that of real numbers. For a normed space Y, we use the notation $\mathbf{B}(Y) = \{y \in Y : \|y\| \leq 1\}$. If Y is an ordered space and $A \subset Y$, we denote by A_+ the positive part of A – that is, $\{a \in A : a \geq 0\}$.

6.2 Benyamini spaces

Here we investigate a class of AM-spaces – the Benyamini spaces. Such spaces are flexible: any separable AM-space can be transformed into a space of this form (Proposition 6.2.3). On the other hand, Benyamini spaces can be easily analyzed, since they have a concrete representation, similar to a $C(K)$ space. In particular, we describe atoms in, and duals of, such spaces in Sections 6.4 and 6.5, respectively.

Definition 6.2.1. *We say that a Banach lattice X is a C-*Benyamini space *(the constant $C > 1$ will often be omitted) if it is a sublattice of $C(K)$, where:*

1. *K is the one point compactification of the union of mutually disjoint compact sets K_n ($K = (\cup_n K_n) \cup \{\infty\}$).*

2. *$X \subset C_0(K)$ – that is, any $x \in X$ vanishes at ∞.*

3. *X separates points for each K_n (that is, for all $t, s \in K_n$, there exists $x \in X$ such that $x(t) \neq x(s)$).*

4. *If $t \in K_m$, $s \in K_n$, and for all $x \in X$, $x(t) = \lambda x(s)$ for some fixed λ, then $\lambda = C^{n-m}$.*

Note that, if X is separable, then each K_n is metrizable (due to (3)). Consequently, K is metrizable.

We begin by establishing some properties of Benyamini spaces.

Lemma 6.2.2. *Let K, C, and $X \subseteq C_0(K)$ be as above. For $n \neq m$, define $D(m,n)$ by*

$$D(m,n) := \{t \in K_m : \exists s \in K_n \text{ such that } \forall x \in X, x(t) = C^{n-m}x(s)\}.$$

Then $D(n,m)$ is closed and homeomorphic to $D(m,n)$.

Proof. Define $\phi_{mn} : D(m,n) \to D(n,m)$ by setting $\phi_{mn}(t)$ to be the unique $s \in D(n,m)$ with the property that $x(t) = C^{n-m}x(s)$ for any $x \in X$. Because X separates points for each K_n, ϕ_{mn} is well defined and injective. By definition of $D(m,n)$, it is bijective, and $\phi_{nm} = \phi_{mn}^{-1}$. To show continuity, suppose $t_k \to t \in D(m,n)$ with $t_k \in D(m,n)$. Suppose there exists a subsequence $s_j = \phi_{mn}(t_{k_j}) \to s \neq \phi_{mn}(t)$ (we can limit ourselves to such a case since K_m is compact). Then for all $x \in X$,

$$x(s) = \lim x(s_j) = \lim_j C^{m-n}x(t_{k_j}) = C^{m-n}x(t) = x(\phi_{m,n}(t)),$$

which is a contradiction, since X separates points. To prove that $D(m,n)$ is closed, suppose $t_k \to t$. Then for all $x \in X$,

$$x(t) = \lim_k x(t_k) = C^{n-m} \lim_k x(\phi_{mn}(t_k)).$$

By compactness, we assume that $\phi_{mn}(t_k) \to s \in K_n$. Hence for all $x \in X, x(t) = C^{n-m}x(s)$, so $s \in D(m,n)$. □

The importance of Benyamini spaces stems from the fact that any separable AM-space can be "approximated" by a Benyamini space.

Proposition 6.2.3. *If X is a separable AM-space, then for every $C > 1$ there exists a Benyamini space X' and a surjective lattice isomorphism $\Phi : X \to X'$ so that for all $x \in X$, $\|x\| \leq \|\Phi(X)\| \leq C\|x\|$.*

The proof below is similar to that of [15, Lemma 1].

Proof. We can assume that $X \subset C(H)$ for some Hausdorff compact H. First, as in [15], we consider the set $F := \cap_{x \in X} x^{-1}(0)$. If $F \neq \emptyset$, identify F with a single point z by passing from K to K/F. Let x_n be a dense sequence in $\mathbf{B}(X)_+$. Let $\psi = (C-1)\sum_{n=1}^{\infty} C^{-n} x_n$. Clearly ψ belongs to X.

Let $H_n = \{t \in H : C^{-n} \le \psi(t) \le C^{-n+1}\}$. If infintely many H_n's are non-empty, let $\widetilde{H}_n$ be disjoint copies of H_n, and let $\widetilde{H} = (\cup_n \widetilde{H}_n) \bigcup \{\infty\}$ be the one point compactification of $\bigcup \widetilde{H}_n$. Otherwise, let $\widetilde{H} = \bigcup_n \widetilde{H_n}$. Define the map $\Psi : \widetilde{H} \to H$ sending $\widetilde{H}_n$ to H_n and ∞ to z. Note that if F is empty, then $\psi(t) > 0$ for all $t \in K$, and since ψ itself is continuous, its image is compact and so must be bounded below; then $H_n = \emptyset$ for n large enough. Otherwise, ψ vanishes only at z. In either case, Ψ is a continuous surjection from $\widetilde{H}$ onto H, which implies that $C(H)$ embeds into $C(\widetilde{H})$ isometrically via the map $x \mapsto \widetilde{\Psi}x := x \circ \Psi$.

Now define a lattice isomorphism $U : C_0(\widetilde{H}) \to C_0(\widetilde{H})$ by setting, for $x \in C_0(\widetilde{H})$, $[Ux](\infty) = 0$, and $[Ux](t) = \dfrac{C^{1-n}x(t)}{(\widetilde{\Psi}\psi)(t)}$. Observe that $\|Ux\| \le \|x\| \le C\|x\|$. Then $T = U \circ \widetilde{\Psi}$ is a lattice homomorphism, and $Y = T(X)$ is a sublattice of $C_0(\widetilde{H})$. We claim that, if $t \in \widetilde{H}_m$ and $s \in \widetilde{H}_n$ are such that $y(t) = \lambda y(s)$ for any $y \in Y'$, then $\lambda = C^{n-m}$. Indeed, $y = Tx$ for some $x \in X$, so

$$\lambda = \frac{y(t)}{y(s)} = \frac{C^{1-m}x(\Psi(t))}{\psi(\Psi(t))} \cdot \frac{\psi(\Psi(s))}{C^{1-n}x(\Psi(s))} = C^{n-m} \cdot \frac{x(\Psi(t))}{x(\Psi(s))} \cdot \frac{\psi(\Psi(s))}{\psi(\Psi(t))}.$$

From this, it follows that $x(\Psi(t))/x(\Psi(s))$ is a constant on X. Either $\Psi(t) = \Psi(s)$, or $t' = \Psi(t)$ and $s' = \Psi(s)$ are "defining points" for $X \subset C(H)$ – that is, $x(t')/x(s')$ is independent of $x \in X$. Either way, $\lambda = C^{n-m}$.

Finally, we transform the sets $\widetilde{H}_n$ into sets K_n, whose points are separated by X'. By the preceding paragraph, if $t, s \in \widetilde{H}_n$ are such that $y(t) = \lambda y(s)$ for any $y \in Y$, then $\lambda = 1$. Define an equivalence relation on $\widetilde{H}$: $t \sim s$ if for all $y \in Y$, $y(s) = y(t)$. Clearly the equivalence classes are closed, hence each quotient space $K_n := \widetilde{H}_n/\sim$ is compact. Identify $\widetilde{H}/\sim$ with $K = (\cup_n K_n) \cup \{\infty\}$, which is the one-point compactification of $\cup_n K_n$. Define $\Phi : Y \to C_0(K)$ by setting, for $y \in Y$, $[\Phi y]([t]) = y(t)$, where $[t]$ is the equivalence class of t. Clearly Φ is a lattice isometry. $X' = \Phi(Y)$ is a Benyamini space, and $\Phi \circ T : X \to X'$ is a lattice isomorphism with desired properties. $\square$

Remark 6.2.4. The Benyamini space X', constructed from X using Proposition 6.2.3, may have a different group of isometries. We do not know whether the Benyamini space can be constructed while preserving the group of isometries (or even a subgroup thereof).

6.3 Extension of functions in Benyamini spaces

We say that a function $x \in C(K_M \cup \ldots \cup K_N)$ is **consistent** if $x(s) = C^{n-m} x(\phi_{mn}(s))$ whenever $s \in D(m, n)$, with $M \leq n, m \leq N$. We shall say that a family of functions $x_n \in C(K_n)$ $(M \leq n \leq N)$ is **consistent** if the function $x \in C(K_M \cup \ldots \cup K_N)$, defined via $x|_{K_n} = x_n$, is consistent.

Proposition 6.3.1. *(1) If $L \leq N$, and $x \in C(K_L \cup \ldots \cup K_N)$ is a consistent function, then there exists $\tilde{x} \in X$ so that $\tilde{x}|_{K_1 \cup \ldots \cup K_N} = x$, and, for $j \notin \{L, \ldots, N\}$, $\sup_{K_j} |\tilde{x}| \leq \max_{L \leq i \leq N} C^{i-j} \sup_{K_i} |x|$.*

(2) If, furthermore, $y \in X_+$ is such that $0 \leq x \leq y$ on $K_L \cup \ldots \cup K_N$, then $\tilde{x}$ can be selected in such a way that, in addition, $0 \leq \tilde{x} \leq y$.

Remark 6.3.2. In a similar fashion, one can show that if $y, z \in X$ are such that $z \leq x \leq y$ on $K_M \cup \ldots \cup K_N$, then $\tilde{x}$ can also be selected in such a way that $z \leq \tilde{x} \leq y$.

The proof of Proposition 6.3.1 is obtained by combining Lemmas 6.3.3 and 6.3.5. First we deal with "downward" extensions.

Lemma 6.3.3. *(1) If $x \in C(K_1 \cup \ldots \cup K_N)$ is a consistent function, then there exists $\tilde{x} \in X$ so that $\tilde{x}|_{K_1 \cup \ldots \cup K_N} = x$, and, for $j > N$, $\sup_{K_j} |\tilde{x}| \leq \max_{1 \leq i \leq N} C^{i-j} \sup_{K_i} |x|$.*

(2) If, furthermore, $y \in X_+$ is such that $0 \leq x \leq y$ on $K_1 \cup \ldots \cup K_N$, then $\tilde{x}$ can be selected in such a way that, in addition, $0 \leq \tilde{x} \leq y$.

Proof. (1) We define $\tilde{x}$ recursively. Suppose $\tilde{x}|_{K_1 \cup \ldots \cup K_{M-1}}$, with $M - 1 \geq N$, has already been defined in such a way that $\sup_{K_j} |\tilde{x}| \leq \max_{1 \leq i \leq N} C^{i-j} \sup_{K_i} |x|$ whenever $N < j < M$. Define now $\tilde{x}$ on K_M. If $t \in D(M, j)$ for some $j < M$, set $x(t) = C^{j-M} x(\phi_{Mj}(t))$. Note that x is well-defined on $\cup_{j<M} D(M, j)$: if $t \in D(M, j) \cap D(M, i)$, then $C^{j-M} x(\phi_{Mj}(t)) = C^{i-M} x(\phi_{Mi}(t))$. Also, for such t, $|x(t)| \leq \max_{1 \leq i \leq N} C^{i-M} \sup_{K_i} |x|$.

Moreover, $\tilde{x}$ is continuous on the closed set $D(M, j)$ for every $j < M$, and thus also on $\cup_{j<M} D(M, j)$. Extend $\tilde{x}$ to a continuous function on K_M without increasing the sup-norm.

Finally, set $\tilde{x}(\infty) = 0$. The function $\tilde{x}$ thusly defined belongs to X. Indeed, it is continuous on each of the sets K_n, and also at ∞, given that $\sup_{K_j} |\tilde{x}| \leq \text{const} \cdot C^{-j}$. Finally, if $t \in D(n, m)$, then $\tilde{x}(t) = C^{m-n} \tilde{x}(\phi_{nm}(t))$.

(2) Modify the recursive process from part (1). Suppose $\tilde{x}|_{K_1 \cup \ldots \cup K_{M-1}}$, where $M - 1 \geq N$, has already been defined in such a way that $0 \leq \tilde{x} \leq y|_{K_1 \cup \ldots \cup K_{M-1}}$ on $K_1 \cup \ldots \cup K_{M-1}$ and $\sup_{K_j} \tilde{x} \leq \max_{1 \leq i \leq N} C^{i-j} \sup_{K_i} x$ whenever $N < j < M$. Define now $\tilde{x}$ on K_M. If $t \in D(M, j)$ for some $j < M$, set $x(t) = C^{j-M} x(\phi_{Mj}(t))$. As before, observe that x is well-defined on $\cup_{j<M} D(M, j)$. Clearly, for $t \in D(M, j)$,

$$0 \leq \tilde{x}(t) \leq y(t), \text{ and } \tilde{x}(t) \leq \max_{1 \leq i \leq N} C^{i-M} \sup_{K_i} x.$$

Also, $\tilde{x}|_{\cup_{j<M} D(M,j)}$ is continuous. Therefore, we can find $u \in C(K_M)$ so that

$$\sup_{K_M} |u| = \sup_{\cup_{j<M} D(M,j)} |\tilde{x}| \leq \max_{1 \leq i \leq N} C^{i-M} \sup_{K_i} |x|.$$

To define $\tilde{x}$ on K_M, set $\tilde{x} = u \wedge y$. $\square$

We shall use the notation $K'_n = K_n \backslash (\cup_{m<n} D(n, m))$, and $K' = \cup_n K'_n$ (note that these sets are open).

In a manner similar to the preceding lemma, one can prove:

Lemma 6.3.4. *Suppose $m \leq n$, $t \in K'_m$, $s \in K'_n$, and $U \subset K'_m$, $V \in K'_n$ are disjoint open sets with the property that $t \in U \subset \overline{U} \subset K'_m$ and $s \in V \subset \overline{V} \subset K'_n$. Then for $\alpha, \beta \in [0, \infty)$, there exists $x \in X_+$ so that:*

1. *For $j < m$, $x|_{K_j} = 0$.*

2. *$x(t) = \alpha$, $x(s) = \beta$, $x \leq \alpha$ on U, and $x \leq \beta$ on V.*

3. *If $m < n$, then $x|_{K_m \backslash U} = 0$.*

4. *If $m < n$, then for $m < j < n$, $0 \leq x|_{K_j} \leq C^{m-j} \alpha$.*

5. *On K_n, $0 \leq x \leq C^{m-n} \alpha \vee \beta$.*

6. *For $j > n$, $0 \leq x|_{K_j} \leq (C^{m-j} \alpha) \vee (C^{n-j} \beta)$.*

Proof. We shall consider the case of $m < n$ (that of $m = n$ is handled similarly). In light of Lemma 6.3.3, it suffices to construct a consistent family of functions $x_j \in C(K_j)$, with $j \leq n$, satisfying the properties listed above. For $j < m$, simply set $x_j = 0$. Define $x_m \in C(K_m)_+$ which vanishes outside of U and satisfies $0 \leq x \leq \alpha = x(t)$.

Use Lemma 6.3.3 to find $x_j \in C(K_j)$ so that the family $(x_j)_{j<n}$ is consistent and $x_j \leq C^{m-j}\alpha$.

Define $x_n \in C(K_n)$ in such a way that:

1. $x_n = 0$ on ∂V, and $0 \leq x_n \leq \beta = x_n(s)$ on V.

2. $x_n(t) = C^{j-n} x_j(\phi_{nj}(t))$ whenever $t \in D(n,j)$ for some $j < n$.

Such a function x_n exists, since $\overline{V}$ is disjoint from $\cup_{j<n} D(n,j)$. Furthermore, the family $(x_j)_{j\leq n}$ is consistent. To define x_j for $j > n$, again invoke Lemma 6.3.3. □

Next we consider "upward" extensions.

Lemma 6.3.5. *(1) If $L \leq N$, and $x \in C(K_L \cup \ldots \cup K_N)$ is a consistent function, then there exists a consistent $\tilde{x} \in C(K_1 \cup \ldots \cup K_N)$ so that $\tilde{x}|_{K_L \cup \ldots \cup K_N} = x$, and for $j < L$, $\sup_{K_j} |\tilde{x}| \leq \max_{L \leq i \leq N} C^{i-j} \sup_{K_i} |x|$.*

(2) If, furthermore, $y \in X_+$ is such that $0 \leq x \leq y$ on $K_L \cup \ldots \cup K_N$, then $\tilde{x}$ can be selected in such a way that, in addition, $0 \leq \tilde{x} \leq y$.

Proof. We only prove (1), as (2) is handled similarly (compare with the proof of Lemma 6.3.3).

Define $\tilde{x}$ recursively. Suppose $\tilde{x}|_{K_{M+1} \cup \ldots \cup K_N}$ $(M+1 \leq L)$ has already been defined in such a way that $\sup_{K_j} |\tilde{x}| \leq \max_{L \leq i \leq N} C^{i-j} \sup_{K_i} |x|$ whenever $M < j < N$. Now define $\tilde{x}$ on K_M. If $t \in D(M,j)$ for some $j \in \{M+1, \ldots, N\}$, set $x(t) = C^{j-M} x(\phi_{Mj}(t))$. Note that x is well-defined on $\cup_{N \leq j < M} D(M,j)$: if $t \in D(M,j) \cap D(M,i)$, then $C^{j-M} x(\phi_{Mj}(t)) = C^{i-M} x(\phi_{Mi}(t))$. Also, for such t, $|x(t)| \leq \max_{1 \leq i \leq N} C^{i-M} \sup_{K_i} |x|$.

As $\tilde{x}|_{\cup_{M<j\leq N} D(M,j)}$ defined above is continuous, we can extend it to the whole K_M, without increasing the sup-norm. □

6.4 Atoms in a Benyamini space

Definition 6.4.1. *A point $k \in K'$ is called* hereditarily isolated *if it is an isolated point of K'_n for some $n \in \mathbb{N}$, and $\phi_{nm}(k)$ is isolated in K_m whenever $k \in D(n,m)$.*

For a point k like this, we can define a function $\theta_k \in X$ by setting $\theta_k(k) = 1$, $\theta_k(\phi_{nm}(k)) = C^{n-m}$ whenever $k \in D(n,m)$, and $\theta_k(t) = 0$ otherwise. Clearly θ_k is a normalized atom in X. Our next result claims that all atoms in X are of this form.

Proposition 6.4.2. *If $x \in X$ is a normalized atom, then $x = \theta_k$ for some hereditarily isolated point k.*

Proof. Suppose $x \in X$ is a normalized atom. Find $k \in K'_n$ such that $x(k) = 1$. We now prove that k is a hereditarily isolated point and that $x = \theta_k$. In particular, we must show that if $k \in D(n,m)$, then $\phi_{nm}(x)$ is isolated in K_m (note that here, $m \geq n$ necessarily).

Suppose, for the sake of contradiction, that $k_m = \phi_{nm}(k)$ is not isolated in K_m for some m. Find the smallest such m. Find distinct $a_1, a_2 \in K_m$ so that $x(a_1), x(a_2) > 1/2$. Find $y \in C(K_m)$ so that $0 \leq y \leq x|_{K_m}$, $y_1(a_1) = \frac{1}{2}$, and $y(a_2) = 0$. By Proposition 6.3.1, there exists $\tilde{y} \in [0, x] \subset X$ such that $\tilde{y}|_{K_m} = y$. By our choice of y, $\tilde{y}$ cannot be a scalar multiple of x. Thus x is not an atom, which is the desired contradiction. $\square$

6.5 The dual of a Benyamini space

Lemma 6.5.1. *Let X and K' be as above. Then X^* is lattice isometric to $M(K')$.*

Proof. Any measure on K' determines a linear functional on X; this gives rise to a contraction $\mathbf{i} : M(K') \to X^*$. We prove that $\mathbf{i}$ is a surjective isometry by showing that any $x^* \in X^*$ can be represented by $\mu \in M(K')$ with $\|\mu\| \leq \|x^*\|$. By the Hahn-Banach Theorem, x^* extends to a functional on $C(K)$ of the same norm; the latter is implemented by a measure $\mu \in M(K)$, with $\|\mu\| = \|x^*\|$. By removing a point mass at ∞, we can and do assume that μ lives on $\cup_n K_n$.

We claim that μ vanishes on $K \backslash K'$. Indeed, otherwise find the smallest value of n for which μ does not vanish on $K_n \backslash K'_n$; then $\mu|_{\cup_{j<n} D(n,j)} \neq 0$. Find the smallest j so that $\mu|_{D(n,j)} \neq 0$. Then the measure

$$\mu' = \mu - \mu\big|_{D(n,j)} + C^{j-n} \mu\big|_{D(n,j)} \circ \phi_{jn}$$

implements the same functional x^*; here, for $x \in C(K)$, we define $\big[\mu\big|_{D(n,j)} \circ \phi_{jn}\big](x)$ to be $\mu\big|_{D(n,j)}\big(x\big|_{D(j,n)} \circ \phi_{nj}\big)$. Note that $\mu'(E) = \mu(E) + C^{j-n}\mu(\phi_{jn}(E))$ for $E \subset D(j,n)$, $\mu'(E) = 0$ for $E \subset D(n,j)$, and $\mu'(E) = \mu(E)$ if E is disjoint from $D(n,j) \cup D(j,n)$. Furthermore, $\mu'|_{K_m} = \mu|_{K_m}$ for $m \notin \{j, n\}$, $\mu'|_{K_n} = \mu|_{K_n \backslash D(n,j)}$, and $\mu'|_{K_j} = \mu|_{K_j} +$

$C^{j-n}\mu\big|_{D(n,j)} \circ \phi_{jn}$. It follows that

$$\|\mu'|_{K_n}\| = \|\mu|_{K_n}\| - \|\mu|_{D(n,j)}\|,$$

while

$$\|\mu'|_{K_j}\| \leq \|\mu|_{K_j}\| + C^{j-n}\|\mu|_{D(n,j)}\|,$$

Therefore,

$$\begin{aligned}\|\mu'\| &= \sum_i \|\mu'|_{K_i}\| = \|\mu'|_{K_n}\| + \|\mu'|_{K_j}\| + \sum_{i \notin \{j,n\}} \|\mu'|_{K_i}\| \\ &\leq (C^{j-n} - 1)\|\mu|_{D(n,j)}\| + \sum_i \|\mu|_{K_i}\| < \sum_i \|\mu|_{K_i}\| = \|x^*\|,\end{aligned}$$

a contradiction.

It is clear that the map $\mathbf{i}$ is positive (a positive measure generates a positive functional). We now show that $\mathbf{i}$ is bipositive: if $\mu \in M(K')$ is not a positive measure, then the corresponding functional is not positive either. We can write $\mu = (\mu_n)$, with (μ_n) concentrated on K_n'. Note that $\|\mu\| = \sum_n \|\mu_n\|$.

Find $N \in \mathbb{N}$ so that $\mu_n \geq 0$ for $n < N$, but μ_N is not positive. By the regularity of the measure μ_N, we can find a positive $x_N \in C(K_N)$, vanishing on $\cup_{j<N} D(N,j)$, so that $\mu_N(x_N) < 0$. By scaling, we can and do assume that $\|x_N\|_\infty = 1$. Let $\delta = -\mu_N(x_N)/3$. Find $M > N$ so that $\sum_{j>M} C^{N-j}\|\mu_j\| < \delta$.

For $j < N$, let x_j be the zero function on K_j. For $N < j \leq M$, find an open set $U_j \subset K_j$ containing $\cup_{i<j} D(j,i)$ with $\|\mu_j|_{U_j}\| < \delta/M$. Now use Lemma 6.3.3 to define, recursively, a consistent family of functions x_j ($j > N$) so that $\|x_j\| \leq C^{N-j}$ and x_j vanishes outside of U_j for $N < j \leq M$. By our choice of U_j, we have $|\mu_j(x_j)| \leq \delta C^{N-j}/M$ for $N < j \leq M$; for $j > M$, we have $|\mu_j(x_j)| \leq \delta C^{N-j}\|\mu_j\|$. Merge all the x_j's into a function $x \in X$. Then

$$\begin{aligned}\mu(x) &\leq \mu_N(x_N) + \sum_{j>N} |\mu_j(x_j)| \leq -3\delta + \sum_{j=N+1}^{M} C^{N-j}\frac{\delta}{M} + \sum_{j>M} C^{N-j}\|\mu_j\| \\ &< -3\delta + (M-N+1)\frac{\delta}{M} + \sum_{j>M} C^{N-j}\|\mu_j\| < -3\delta + \delta + \delta = -\delta,\end{aligned}$$

which shows that the linear functional determined by μ is not positive.

We have established that $\mathbf{i} : M(K') \to X$ is a bipositive surjective isometry. By [2],

$\mathbf{i}$ is a lattice isometry. $\square$

We shall denote by $\mathcal{A}_1$ the set of normalized atoms of X^*. By Lemma 6.5.1, $X^* = M(K')$, hence $\mathcal{A}_1 = \{\delta_t : t \in K'\} \subset \mathbf{B}(X^*)_+$. Below we show that $\mathcal{A}_1$ (equipped with the weak* topology inherited from X^*) is topologically homeomorphic to K'.

Lemma 6.5.2. *The map* $\mathbf{j} : K' \to \mathcal{A}_1 : t \mapsto \delta_t$ *is a homeomorphism.*

Proof. To establish the continuity of $\mathbf{j}$, suppose the net t_α converges to t in K'. By continuity, $\delta_{t_\alpha}(x) = x(t_\alpha) \to x(t) = \delta_t(x)$ for any $x \in X$, hence $\delta_{t_\alpha} \to \delta_t$ in the weak* topology.

For the continuity of $\mathbf{j}^{-1}$, consider a net $(t_\alpha) \subset \mathcal{A}_1$ so that $\delta_{t_\alpha} \to \delta_t \in \mathcal{A}_1$ in the weak* topology – that is, $x(t_\alpha) \to x(t)$ for any $x \in X$. By the compactness of K, it suffices to show that the limit of any convergent subnet of (t_α) is t.

Suppose (t'_β) is a subnet of (t_α), which converges to $s \in K$. Then for any $x \in X$, we have $x(s) = \lim_\beta x(t'_\beta) = x(t)$. As $x(t)$ is not always 0, part (2) of Definition **??** implies $s \neq \infty$. Further, $x(t) = x(s)$ for any $x \in X$, hence parts (3) and (4) of Definition **??** show that $t = s$.

$\square$

6.6 Renormings of Benyamini spaces

Theorem 6.6.1. *Suppose* $(X, \|\cdot\|)$ *is a Benyamini space. Then, for any* $c > 1$, X *can be equipped with an equivalent norm* $|\!|\!|\cdot|\!|\!|$ *so that* $\|\cdot\| \leq |\!|\!|\cdot|\!|\!| \leq c^2\|\cdot\|$, *so that the identity is the only lattice isometry on* $(X, |\!|\!|\cdot|\!|\!|)$. *If* X *has no more than one atom, then* $|\!|\!|\cdot|\!|\!|$ *can be selected to be an AM-norm.*

Remark 6.6.2. The restriction on the number of atoms is essential here; see Remark 6.6.8.

The rest of this section is devoted to proving Theorem 6.6.1.

Assume that X is a C-Benyamini space ($C < 2$) and that $c < \sqrt[3]{C}$. Let A and B be the sets of all $n \in \mathbb{N}$ for which K'_n is infinite, resp. finite and non-empty. For $n \in B$, write $K'_n = \{t_{1n}, \ldots, t_{p_n n}\}$. For $n \in A$, find a sequence $t_{1n}, t_{2n}, \ldots$ of distinct elements of K'_n which is dense in K'_n. Find a family $(\lambda_{in})_{n \in A \cup B} \subset (1, c)$ of distinct numbers so that: (i) for $n \in A$, $c > \lambda_{1n} > \lambda_{2n} > \ldots$, and $\lim_i \lambda_{in} = 1$; (ii) for $n \in B$,

$c > \lambda_{1n} > \ldots > \lambda_{p_n n} > 1$. For each $t \in K'$, let $\mu(t) = \lambda_{in}$ if $t = t_{in}$ for some i and n, $\mu(t) = 1$ otherwise.

Denote the normalized atoms of X by $(\theta_i)_{i \in I}$, where the set I is countable. By Proposition 6.4.2, each θ_i corresponds with a hereditarily isolated point $a_i \in K'$. Furthermore, for each i, there exists a canonical band projection P_i onto $\operatorname{span}[\theta_i]$. Then $P_i x = x(a_i)\theta_i$.

Our definition of $|||\cdot|||$ would depend on the cardinality of I.

$\underline{|I| = 0}$. For $x \in X$ set

$$|||x||| = \sup_{t \in K'} \mu(t)|x(t)|. \tag{6.1}$$

$\underline{|I| = 1}$. Write $I = \{1\}$; represent X as $X_1 \oplus \mathbb{R}$, where $X_1 = \ker P_1$ is a C-Benyamini space (with the underlying space obtained by removing from K all the points $\phi_{nm}(a_1)$, when $a_1 \in K_n$ and $m \geq n$). Let $|||\cdot|||_1$ be the norm defined on X_1 using (6.1) (with some collection (t_{ni})). Let

$$|||x||| = \max\left\{|||(I - P_1)x|||_1, \|P_1 x\|\right\}. \tag{6.2}$$

$\underline{|I| > 1}$. Write $I = \{1, \ldots, m\}$ $(2 \leq m < \infty)$ or $I = \mathbb{N}$. Let $\mathcal{P} = \{(i,j) \in I^2 : i < j\}$, and let $\pi : \mathcal{P} \to \mathbb{N}$ be an injection. For $(i,j) \in \mathcal{P}$, let $\| \cdot \|_{i,j}$ be the norm on $\mathbb{R}^2$ whose unit ball is an octagon with vertices

$$\left(\pm\left(1 - \frac{c-1}{c(2\pi(i,j)+1)}\right), \pm 1\right) \text{ and } \left(\pm 1, \pm\left(1 - \frac{c-1}{2c\pi(i,j)}\right)\right)$$

We mention some properties of the norms $\| \cdot \|_{i,j}$, to be used in the sequel.

N1 $\| \cdot \|_\infty \leq \| \cdot \|_{i,j} \leq c\| \cdot \|_\infty$.

N2 The formal identity $(\mathbb{R}^2, \| \cdot \|_{i_1,j_1}) \to (\mathbb{R}^2, \| \cdot \|_{i_2,j_2})$ (with the first vector of the canonical basis mapping to the first, and the second – to the second) is an isometry iff $i_1 = i_2$ and $j_1 = j_2$. This follows from a comparison of extreme points.

N3 For $\gamma > 1$ and $k \in I$, there exists $L = L(k, \gamma) \geq k$ so that $\| \cdot \|_{k,j} \leq \gamma \| \cdot \|_\infty$ for $j > L$.

N4 For $\gamma > 1$, there exists $M = M(\gamma)$ so that $\| \cdot \|_{i,j} \leq \gamma \| \cdot \|_\infty$ whenever $j > i > M$.

N5 If $|\alpha| \vee |\beta| = 1$ and $|\alpha| \wedge |\beta| \leq 1/c$, then $\|(\alpha, \beta)\|_{ij} = 1$.

We let

$$|||x||| = \max\left\{ \sup_{t \in K'} \mu(t)|x(t)|, \sup_{(i,j) \in \mathcal{P}} \left\| \left(\mu(a_i)x(a_i), \mu(a_j)x(a_j)\right) \right\|_{i,j} \right\}. \tag{6.3}$$

Clearly, we always have $\|\cdot\| \leq |||\cdot||| \leq c^2\|\cdot\|$ (in fact, if $|I| \leq 1$, we can replace c^2 by c). It is also clear that for $|I| \leq 1$, $|||\cdot|||$ is an AM-norm. To show that the only lattice isometry on $(X, |||\cdot|||)$ is the trivial one, we need a series of lemmas. As the proof for $|I| = 1$ follows immediately from that for $|I| = 0$, we shall only consider the cases of $I = \emptyset$ and $|I| \geq 2$.

First we establish the norms of point masses. Let $\hat{\delta}_t = \mu(t)\delta_t$.

Lemma 6.6.3. *For any $t \in K'$, $\left|\left|\left|\hat{\delta}_t\right|\right|\right| = 1$.*

Proof. For $x \in X$ and $t \in K'$, we clearly have $|||x||| \geq \mu(t)|x(t)| = \left|\hat{\delta}_t(x)\right|$, hence $\left|\left|\left|\hat{\delta}_t\right|\right|\right| \leq 1$. It remains to prove the opposite inequality.

Fix $t \in K'$ and $\gamma > 1$. We need to find $x \in X_+$ such that $x(t) = 1/\mu(t)$ and $|||x||| \leq \gamma$. To this end, find n so that $t \in K'_n$. Next, construct a finite set $V \subset K'_n$ consisting of "potentially troublemaking" points. If $|I| = \emptyset$, let

$$V = \{s \in K'_n : \mu(s) > \gamma\mu(t)\}.$$

If $|I| \geq 2$ and t is not hereditarily isolated, let

$$V = \{s \in K'_n : \mu(s) > \gamma\mu(t)\} \cup \{a_i \in K'_n : i \leq M(\gamma)\},$$

with $M(\gamma)$ as in [N4].

If $|I| \geq 2$ and t is hereditarily isolated, then $t = a_k$ for some k. Let

$$V = \{s \in K'_n : \mu(s) > \gamma\mu(t)\} \cup \{a_i \in K'_n : i \leq M(\gamma) \vee L(k, \gamma)\} \backslash \{a_k\},$$

where $L(k, \gamma)$ comes from property [N3].

The set V is finite and does not contain t. Find an open set $U \subset K'_n \backslash V$ containing t. Find $x \in C(K_n)$ such that x vanishes outside of U and $0 \leq x \leq 1/\mu(t) = x(t)$. Define x to be 0 on K_m for $m < n$. This function is consistent, so by Proposition 6.3.1, there exists $\tilde{x} \in X_+$ so that $\tilde{x}|_{K_1 \cup \ldots \cup K_n} = x$ and $\|\tilde{x}\| = 1/\lambda_{in}$.

It remains to show that $|||\tilde{x}||| \leq \gamma^2$. This will follow if we establish that

$$\mu(s)|\tilde{x}(s)| \leq \gamma \text{ for any } s \in K', \tag{6.4}$$

and (in the case of $|I| \geq 2$)

$$\big\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\big\|_{i,j} \leq \gamma^2 \text{ for any } i < j. \tag{6.5}$$

Note that, due to our construction of $\tilde{x}$, $\tilde{x}(s) = 0$ if $s \in K'_m$ with $m < n$. For $s \in K'_n$, we have $\tilde{x}(s) = 0$ for $s \notin U$, while for $s \in U$, $\mu(s) \leq \gamma\mu(t)$, so $\mu(s)|\tilde{x}(s)| \leq \gamma$. Finally, if $s \in K'_m$ for some $m > n$, we have $\tilde{x}(s) \leq C^{n-m}/\mu(t)$, hence $\mu(s)|\tilde{x}(s)| \leq c/C < 1 < \gamma$. This establishes (6.4).

To handle (6.5), note that if $a_i \in \cup_{m<n} K'_m \cup (K'_n \backslash U)$, then $\tilde{x}(a_i) = 0$, and therefore,

$$\big\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\big\|_{i,j} = \big\|\big(0, \mu(a_j)\tilde{x}(a_j)\big)\big\|_{i,j} = \mu(a_j)\tilde{x}(a_j).$$

The right hand side cannot exceed γ, as discussed in the paragraph relating to (6.4). The same conclusion holds if $a_j \in \cup_{m<n} K'_m \cup (K'_n \backslash U)$.

If $a_i, a_j \in \cup_{\ell>n} K'_\ell$, then $\tilde{x}(a_i), \tilde{x}(a_j) \leq 1/(\mu(t)C)$, hence

$$\big\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\big\|_{i,j} \leq \frac{c^2}{\mu(t)C} < 1.$$

Now consider the case of $a_i \in U$, $a_j \in \cup_{\ell>n} K'_\ell$. In this situation, $\mu(a_j)\tilde{x}(a_j) < c/C < c^{-2}$, hence, by [N5],

$$\big\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\big\|_{i,j} \leq \gamma.$$

The same conclusion holds if $a_j \in U$, $a_i \in \cup_{\ell>n} K'_\ell$.

Finally, if $a_i, a_j \in U$, then $\mu(a_i), \mu(a_j) \leq \gamma\mu(t)$. By the choice of U,

$$\big\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\big\|_{i,j} \leq \gamma\big\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\big\|_\infty \leq \gamma^2.$$

The same conclusion holds if the roles of a_i and a_j are reversed. We have now established (6.5). $\square$

Now suppose T is a surjective lattice isometry on $(X, |||\cdot|||)$. Note first that T fixes the atoms of X:

Lemma 6.6.4. *For any $i \in I$, $T\theta_i = \theta_i$.*

Proof. This is obvious if $|I| \leq 1$. For $|I| \geq 2$, let $e_i = \theta_i/\mu(a_i)$ be the normalized atoms. By (6.3), for any $\alpha, \beta \in \mathbb{R}$, we have

$$\|\!|\alpha e_i + \beta e_j|\!\|= \big\|(\alpha, \beta)\big\|_{i,j}.$$

If T maps e_i and e_j to e_k and e_ℓ respectively, then

$$\big\|(\alpha, \beta)\big\|_{i,j} = \big\|(\alpha, \beta)\big\|_{k,\ell} \text{ for any } \alpha, \beta,$$

which, in light of Property [N2], implies $i = k$, $j = \ell$. □

Now observe that T^* is interval preserving [61, Theorem 1.4.19], hence it maps atoms of X^* to atoms. The atoms in X^* are characterized by Proposition 6.4.2. By Lemma 6.6.3, the set of normalized atoms of $(X^*, \|\!|\cdot|\!\|)$ (which we shall denote by $\mathcal{A}$) coincides with $\{\hat{\delta}_t : t \in K'\}$.

Thus, by Lemma 6.6.3, there exists a bijection $\psi : K' \to K'$ so that $T^*\hat{\delta}_t = \hat{\delta}_{\psi(t)}$. We shall show that $\psi(t) = t$ is the identity map. In fact, Lemma 6.6.4 already shows that $\psi(t) = t$ if t is a hereditarily isolated point.

To proceed further, in the next few lemmas we examine weak* convergence in $\mathcal{A}$. For convenience, we denote by ϕ_{nn} the identity map on $D(n, n) := K_n$.

Lemma 6.6.5. *Suppose $m, n \in \mathbb{N}$, $t \in K'_n$, and the sequence $(t_i) \subset K'_m \backslash \{t\}$ converges to s. Then the following are equivalent:*

1. *$m \geq n$, and $s = \phi_{nm}(t)$.*

2. *$\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \alpha\hat{\delta}_t$ for some $\alpha > 0$.*

Moreover, if (1) holds, then (2) holds with $\alpha = C^{n-m}/\mu(t)$.

Proof. To show that (1) implies (2), as well as the "moreover" statement, we only need to observe that, due to our selection of (λ_{jm}), we have $\lim_i \mu(t_i) = 1$. We need to establish the converse.

First show that $m \geq n$. If $m < n$, then find an open set $U \subset K'_n$ containing t. By Proposition 6.3.1, there exists $x \in X$ so that $0 \leq x \leq 1 = x(t)$, which vanishes on

$K_n \backslash U$ and on K_j for $j < n$. In particular, $\hat{\delta}_t(x) \neq 0$, while $\hat{\delta}_{t_i}(x) = 0$ for any i. This contradicts (2).

Thus $m \geq n$. Next show that $t \in D(n,m)$ and $s = \phi_{nm}(t)$. Suppose, for the sake of contradiction, that either $t \notin D(n,m)$, or $t \in D(n,m)$ and $s \neq \phi_{nm}(t)$. Find the smallest $i \leq m$ so that $s \in D(m,i)$, and let $s' = \phi_{mi}(s)$. Then $t \neq s'$. By Lemma 6.3.4, there exists $x \in X$ so that $x(t) = 1$ and $x(s') = 0$, hence also $x(s) = 0$. We observe that $\hat{\delta}_t(x) \neq 0$ and $\lim_i \hat{\delta}_{t_i}(x) = 0$, again contradicting (2). □

Lemma 6.6.6. *Suppose we are given $t \in K'_n$ and a sequence $(t_i) \subset K' \backslash \{t\}$. Then the following are equivalent:*

1. *There exists $m \geq n$ so that for i large enough, $t_i \in K'_m$. Furthermore, (t_i) converges to $s = \phi_{nm}(t)$.*

2. $\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \alpha \hat{\delta}_t$ *for some $\alpha > 0$.*

Moreover, if (1) holds, then, in (2), $\alpha = C^{n-m}/\mu(t)$.

Proof. Lemma 6.6.5 shows that (1) implies (2), as well as the "moreover" conclusion. To establish (2) ⇒ (1), find, for each i, $m(i) \in \mathbb{N}$ so that $t_i \in K'_{m(i)}$. We shall show that the sequence $(m(i))$ is eventually constant.

First we show that $(m(i))$ is bounded. Indeed, otherwise we can find a sequence (i_p) so that $\lim_p m(i_p) = \infty$. Clearly $\lim x(t_{i_p}) = 0$ for any $x \in X$, hence $\hat{\delta}_{i_p} \xrightarrow{w^*} 0$.

Now suppose, for the sake of contradiction, that $(m(i))$ does not stabilize. Passing to a subsequence, we can assume that there exists $m_1 \neq m_2$ so that $m(i) = m_1$ if i is odd, and $m(i) = m_2$ is even if i is even. Further, we can assume that (t_{2i-1}) and (t_{2i}) converge to $s_1 \in K_{m_1}$ and $s_2 \in K_{m_2}$, respectively. From Lemma 6.6.5, $m_1, m_2 \geq n$, $t_{2i} \to s_2 = \phi_{m_2 n}(t)$, and $\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \hat{\delta}_t/(C^{m_2-n}\mu(t))$. Similarly, $t_{2i-1} \to s_1 = \phi_{m_1 n}(t)$, and $\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \hat{\delta}_t/(C^{m_1-n}\mu(t))$. Thus, $1/\alpha = C^{m_2-n}\mu(t) = C^{m_1-n}\mu(t)$, which leads to the impossible conclusion $m_1 = m_2$.

Thus, the sequence $(m(i))$ is eventually constant. To conclude the proof, invoke Lemma 6.6.5. □

Lemma 6.6.7. *Suppose $t \in K'$ is not hereditarily isolated. Then there exists a sequence $(t_i) \subset K'$ so that $\hat{\delta}_{t_i} \xrightarrow{w^*} \alpha \hat{\delta}_t$, for some $\alpha \in (0,1]$. Moreover, for every such sequence there exists $r \in \{0,1,2,\ldots\}$ so that $\alpha = 1/(C^r \mu(t))$.*

Proof. Suppose first t is not isolated in K_n. Then t cannot be isolated in the open subset $K'_n \subset K$, so we can find a sequence $(t_i) \subset K'_n$, converging to t. Clearly $\delta_{t_i} \to \delta_t$ (in the weak* topology). Moreover, $\mu(t_i) \to 1$, hence $\hat{\delta}_{t_i} \to \alpha\hat{\delta}_t$, where $\alpha = 1/\mu(t) \in (1/c, 1]$.

Now suppose t is isolated in K_n (equivalently, in K'_n). Use Proposition 6.4.2 to find the smallest $m > n$ s.t. $s = \phi_{nm}(t)$ is not isolated in K_m. We claim that K'_m is non-empty, and s belongs to the closure. Indeed, as $t \in K'_n$, s cannot belong to $D(m,k)$ with $k < n$. In addition, if $s \in D(m,k)$ for some $n \leq k < m$, then s is an isolated point of $D(m,k)$, due to the minimality of m. Consequently, s is an isolated point of $\cup_{k<m} D(m,k)$. As s is not isolated in K_m, we can find a sequence $(t_i) \subset K'_m$ converging to t. Then $\delta_{t_i} \stackrel{w^*}{\to} C^{n-m}\delta_t$, hence $\hat{\delta}_{t_i} \to \alpha\hat{\delta}_t$, where $\alpha = C^{n-m}/\mu(t) \in (C^{n-m}/c, C^{n-m}]$.

Now suppose $\hat{\delta}_{t_i} \stackrel{w^*}{\to} \alpha\hat{\delta}_t$, for some $\alpha \in (0,1]$. By Lemma 6.6.6, there exists m so that $t_i \in K_m$, for m large enough; and furthermore, $t_i \to \phi_{mn}(t)$. As in the previous paragraph, $\alpha = C^{n-m}/\mu(t)$. $\square$

Theorem 6.6.1 – completion of the proof. Suppose T is a lattice isometry on $(X, |||\cdot|||)$. By Section 6.5, it suffices to show that $T^*\hat{\delta}_t = \hat{\delta}_t$ for any $t \in K'$. As T^* maps normalized atoms to normalized atoms, $T^*\hat{\delta}_t = \hat{\delta}_s$, where $s = \psi(t) \in K'$. By Lemma **??**, $\psi(t) = t$ if t is hereditarily isolated. As the set $\mathcal{A}$ of normalized atoms is identified with $\left\{\hat{\delta}_t : t \in K'\right\}$, we conclude that t is not hereditarily isolated iff $\psi(t)$ satisfies the same condition. For future use, note that if t is hereditarily isolated, then $t = t_{in}$ for some i, n.

Now suppose t is not hereditarily isolated. Let $s = \psi(t)$. In light of Lemma 6.6.7, there exists a sequence $(u_i) \subset K'$ so that $\hat{\delta}_{u_i} \stackrel{w^*}{\to} \alpha\hat{\delta}_t$. Moreover, for every such sequence,

$$\frac{1}{\mu(t)} = \nu(t) := \sup\left\{C^k\alpha : k \in \{0,1,2,\ldots\}, C^k\alpha \leq 1\right\}.$$

Being isometric and weak* to weak* continuous, T^* preserves $\nu(\cdot)$, hence $\mu(\psi(t)) = \mu(t)$, for any $t \in K'$.

Recall that t_{in} is the unique point t with $\mu(t) = \lambda_{in}$. Consequently, $\psi(t_{in}) = t_{in}$, or equivalently, $T^*\hat{\delta}_{t_{in}} = \hat{\delta}_{t_{in}}$.

Now suppose $t \in K'\backslash(\cup_{i,n}\{t_{in}\})$ is not hereditarily isolated. Find a sequence $\left(t_{i_j n_j}\right)_j$ which converges to $\phi_{mn}(t)$ for some $m \geq n$. By Lemma 6.6.5,

$$\mathrm{w}^* - \lim_j \hat{\delta}_{t_{i_j n_j}} = C^{n-m}\hat{\delta}_t,$$

hence, due to the weak* to weak* continuity of T^*,

$$\mathrm{w}^* - \lim_j T^* \hat{\delta}_{t_{i_j n_j}} = C^{n-m} \hat{\delta}_{\psi(t)},$$

However, the left hand sides of the two centered expressions coincide, hence $\psi(t) = t$. $\square$

Remark 6.6.8. In Theorem 6.6.1, the desired renorming cannot be an AM-space if the number of atoms exceeds 1. Indeed, suppose $a_1, \ldots, a_n$ are normalized atoms in an AM-space X, and let $X_0 = \{a_1, \ldots, a_n\}^\perp$. If π is a permutation of $\{1, \ldots, n\}$, then $T : X \to X$, defined by $Ta_i = a_{\pi(i)}$ and $Tx = x$ for $x \in X_0$, is an isometry. Thus, any AM renorming of a space with more than one atom will have non-trivial lattice isometries.

6.7 Questions and further research

Question 6.7.1 (Conditions for displays in lattice automorphism groups)**.** The work in this chapter is part of a larger, on-going project with Oikhberg on Banach lattice analogues of the displayability of Polish groups. Let E be a lattice, let $ISO_L(E)$ be the group of lattice isometries over E, and suppose G is a Polish group. Then we say that G **is displayable in** E if E has an equivalent lattice norm $\vert\!\vert\!\vert\cdot\vert\!\vert\!\vert$ such that there is a homeomorphic group isomorphism between G and $ISO_L(E, \vert\!\vert\!\vert\cdot\vert\!\vert\!\vert)$ (Let us denote this by $G \approxeq ISO_L(E, \vert\!\vert\!\vert\cdot\vert\!\vert\!\vert)$). This lead to the following general question: Under what conditions on G and E is G displayable in E?

To this end, we consider three specific questions that have arisen in ongoing work.

Question 6.7.2 (Displayability over LUR lattices)**.** In [36] and [33], the authors show that certain groups under mild conditions can be displayed on separable Banach spaces (here, displayability is with respect merely to linear isometries) unlike with Banach lattices). One of the assumptions is that the Banach space X has an **locally uniformly rotund (LUR) norm**, meaning that for any point $x \in \mathbf{S}(X)$ and for all $\varepsilon > 0$, there exists $\delta > 0$ so that for all $y \in \mathbf{S}(X)$, if $\|x - y\| \geq \varepsilon$, then $\|x + y\| \leq 2 - \delta$. Intuitively, local uniform rotundness captures the absence of anything resembling corners and straight edges in the unit ball of a space. Another assumption is on G and its orbits. a point $x \in X$ is **distinguished by G** if

$$\inf_{g \in G \setminus \{Id\}} \|x - gx\| > 0.$$

If X both has an LUR norm and a point x distinguished by G, then G is displayable in X. The approach that is a "group invariant" version of Bellenot's method in [11]: take a the unit ball of the space and carefully add "pimples" of varying size to the unit ball, but which are invariant under the isometries in G. Because the pimples comprise all the "corners" of this new unit ball (which originally had no such corners, because X is LUR), any isometry would have to map pimples to pimples.

Can a similar approach can be employed with lattices? A theorem by Kadec in [45] shows that any separable Banach space has an equivalent LUR norm. However, a result from [23] shows that a Banach lattice has an equivalent LUR *lattice norm* iff it is order continuous. Thus a method like this would be limited to order continuous lattices. One would also need to make pimples in a way that the resulting unit ball of a renormed lattice E is also solid, in addition to being closed.

Question 6.7.3 (Displayability over $C_0(X)$ spaces)**.** For non-order continuous lattices, a different method is to be used. We consider the class of $C_0(X)$ spaces, with X locally compact Polish. Here, an expansion of the technique used for AM-spaces is useful, since one can carefully weigh points, but keep the weights invariant under lattice isometries as well.

Question 6.7.4 (Restrictions to displayability)**.** Not every group can be displayed in a lattice X. For Banach spaces, some clear restrictions on potential displayable groups are that they must be SOT-closed and that they must include $-Id$. Another limitation is that of [36, Proposition 4.iii]: displayable groups over a given Banach space X cannot be convex transitive. A linear isometry group G is **convex transitive** on X if for all $x \in \mathbf{S}(X), \overline{CH}(G_x) = \mathbf{B}(X)$. One also says that a norm on X is convex transitive if the group of linear isometries $ISO(X)$ on X is convex transitive. This necessary condition shows that the proper displayable subgroups of the linear isometry group on X must be "thin" in some sense. The proof of this proposition echoes that of Cowie's proof in [22] that a Banach space X has a convex transitive norm $\|\cdot\|$ iff it is **uniquely maximal**, meaning that it is not properly contained in any bounded subgroup of linear isomorphisms and any equivalent norm $|||\cdot|||$ with the same group of isometries must a linear multiple of $\|\cdot\|$.

What sorts of restrictions are there on displayable groups in the Banach lattice setting? Clearly, lattice isometries cannot be convex transitive in the way that linear isometries are, because lattice isometries must be positive maps. One can frame an analogue of the restriction from convex transitivity in terms of lattice isometries. We say that a group $G \subseteq ISO_L(E)$ of lattice isometries over E is **solid convex transitive** if for all $x \in \mathbf{S}(E)_+$, we have $\mathrm{CSCH}(G_x) = \mathbf{S}(E)$. We also employ a similar definition for unique maximality, except instead of bounded linear isomorphism or linear isometry groups and Banach space norms, we have lattice isomorphism and lattice isometry groups and Banach lattice norms where appropriate. One can then ask: is unique maximality for Banach lattices equivalent to solid convex transitivity? Furthermore, is it the case that proper displayable subgroups of $ISO_L(E)$ are not solid convex transitive?

References

[1] Abramovich, Y. A., and Aliprantis, C. D. *An invitation to operator theory*, vol. 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] Abromovich, Y. Isometries of normed lattices (in Russian). *Optimizatsiya 43(60)* (1988), 74–80.

[3] Aliprantis, C., and Burkinshaw, O. *Positive operators*, vol. 119. Springer, 2006.

[4] Amarante, M. The Sandwich theorem via Pataraia's fixed point theorem. *Positivity 23*, 1 (2019), 97–100.

[5] Anisca, R., Ferenczi, V., and Moreno, Y. On the classification of positions and complex structures in Banach spaces. *J. Funct. Anal. 272*, 9 (2017), 3845–3868.

[6] Argyros, S. A., Godefroy, G., and Rosenthal, H. P. Descriptive set theory and Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*. North-Holland, Amsterdam, 2003, pp. 1007–1069.

[7] Avilés, A., Cabello, F., Castillo, J. M. F., González, M., and Moreno, Y. Banach spaces of universal disposition. *J. Funct. Anal 261* (2011), 2347–2361.

[8] Avilés, A., and Tradacete, P. Amalgamation and injectivity in Banach lattices. *arXiv:2007.15261* (2020).

[9] Banakh, T., and Garbulińska-Wegrzyn, J. The universal Banach space with a K-suppression unconditional basis. *arXiv preprint arXiv:1801.07433* (2018).

[10] Banakh, T., Garbulińska-Wgrzyn, J., et al. A universal Banach space with a K-unconditional basis. *Advances in Operator Theory 4*, 3 (2019), 574–586.

[11] Bellenot, S. Banach spaces with trivial isometries. *Israel J. Math 56* (1986), 89–96.

[12] Ben Yaacov, I. The linear isometry group of the Gurarij space is universal. *Proceedings of the American Mathematical Society 142*, 7 (2014), 2459–2467.

[13] Ben Yaacov, I. Fraïssé limits of metric structures. *J. Symb. Log. 80*, 1 (2015), 100–115.

[14] Ben Yaacov, I., Berenstein, A., Henson, C. W., and Usvyatsov, A. Model theory for metric structures. In *Model theory with applications to algebra and analysis. Vol. 2*, vol. 350 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 2008, pp. 315–427.

[15] Benyamini, Y. Separable G spaces are isomorphic to $C(K)$ spaces. *Israel J. of Math. 14*, 3 (1973), 287–293.

[16] Bossard, B. A coding of separable Banach spaces. analytic and coanalytic families of Banach spaces. *Fund. Math. 172*, 2 (2002), 117–152.

[17] Bourgain, J., and Talagrand, M. Dans un espace de Banach reticulé solide, la propriété de Radon-Nikodỳm et celle de Kreĭn–Milman sont équivalentes. *Proceedings of the American Mathematical Society 81*, 1 (1981), 93–96.

[18] Bourgin, R. D. *Geometric aspects of convex sets with the Radon-Nikodym property*, vol. 993. Springer, 2006.

[19] Cartwright, D. I. *Extensions of positive operators between Banach lattices.* No. 3-164. Amer Mathematical Society, 1975.

[20] Caselles, V. A short proof of the equivalence of KMP and RNP in Banach lattices and preduals of von Neumann algebras. *Proc. Amer. Math. Soc. 102*, 4 (1988), 973–974.

[21] Chen, Z., and Wickstead, A. Relative weak compactness of solid hulls in Banach lattices. *Indagationes Mathematicae 9*, 2 (1998), 187–196.

[22] Cowie, E. A note on uniquely maximal Banach spaces. *Proceedings of the Edinburgh Mathematical Society 26*, 1 (1983), 85–87.

[23] Davis, W., Ghoussoub, N., and Lindenstrauss, J. A lattice renorming theorem and applications to vector-valued processes. *Transactions of the American Mathematical Society 263* (1981), 531–540.

[24] Diestel, J., et al. Geometry of Banach spaces- selected topics.

[25] Diestel, J., and Uhl Jr, J. Vector measures. *Bull. Am. Math. Soc 84* (1978), 681–685.

[26] Ding, L. On equivalence relations generated by Schauder bases. *J. Symb. Log. 82*, 4 (2017), 1459–1481.

[27] Dougherty, R., and Hjorth, G. Reducibility and nonreducibility between l^p equivalence relations. *Trans. Amer. Math. Soc. 351*, 5 (1999), 1835–1844.

[28] Eagle, C. J., Farah, I., Hart, B., Kadets, B., Kalashnyk, V., and Lupini, M. Fraïssé Limits of C^*-Algebras. *J. Symb. Log. 81* (2016), 755–773.

[29] Eggleston, H. G. *Convexity.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 47. Cambridge University Press, New York, 1958.

[30] Elliott, G. A., Farah, I., Paulsen, V. I., Rosendal, C., Toms, A. S., and Törnquist, A. The isomorphism relation for separable C*-algebras. *Math. Res. Lett. 20*, 6 (2013), 1071–1080.

[31] Fabian, M., Habala, P., Hájek, P., Montesinos, V., and Zizler, V. *Banach space theory: the basis for linear and nonlinear analysis.* Springer Science & Business Media, 2011.

[32] Fabian, M., Habala, P., Hájek, P., Santalucía, V. M., Pelant, J., and Zizler, V. *Functional analysis and infinite-dimensional geometry.* Springer Science & Business Media, 2013.

[33] Ferenczi, V., and Galego, E. Countable groups of isometries on Banach spaces. *Trans. Amer. Math. Soc. 362*, 8 (2010), 4385–4431.

[34] Ferenczi, V., Lopez-Abad, J., Mbombo, B., and Todorcevic, S. Amalgamation and Ramsey properties of L_p spaces. *Adv. Math. 369* (2020), 107190, 76.

[35] Ferenczi, V., Louveau, A., and Rosendal, C. The complexity of classifying separable Banach spaces up to isomorphism. *J. Lond. Math. Soc. (2) 79*, 2 (2009), 323–345.

[36] Ferenczi, V., and Rosendal, C. Displaying Polish groups on separable Banach spaces. *Extracta Math. 2* (2011), 195–233.

[37] Ferenczi, V., and Rosendal, C. On isometry groups and maximal symmetry. *Duke Mathematical Journal 162*, 10 (2013), 1771–1831.

[38] Fuchssteiner, B., and Wright, M. Representing isotone operators on cones. *The Quarterly Journal of Mathematics 28*, 2 (1977), 155–162.

[39] Gao, S., and Kechris, A. S. *On the classification of Polish metric spaces up to isometry.* American Mathematical Soc., 2003.

[40] Ghasemi, S., and Kubiś, W. Universal AF-algebras. *Journal of Functional Analysis* (2020), 108590.

[41] Godefroy, G. The isomorphism classes of l_p are Borel. *Houston J. Math. 43*, 3 (2017), 947–951.

[42] Harrington, L. A., Kechris, A. S., and Louveau, A. A Glimm-Effros dichotomy for Borel equivalence relations. *Journal of the American mathematical society* (1990), 903–928.

[43] Hjorth, G. *Classification and Orbit Equivalence Relations*, vol. 75 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2000.

[44] Jarosz, K. Any banach space has an equivalent norm with trivial isometries. *Israel Journal of Mathematics 64*, 1 (1988), 49–56.

[45] Kadec, M. On spaces isomorphic to locally uniformly rotund spaces. *Izv. Vyzs. Ycz. Zav. Mat.(8) 1* (1959), 51–57.

[46] Kechris, A. S. *Classical Descriptive Set Theory*, vol. 156 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[47] Kubiś, W. Fraïssé sequences: category-theoretic approach to universal homogeneous structures. *Ann. Pure Appl. Logic 165*, 11 (2014), 1755–1811.

[48] Kubiś, W., and Shelah, S. Homogeneous structures with nonuniversal automorphism groups. *The Journal of Symbolic Logic 85*, 2 (2020), 817–827.

[49] Kubiś, W., and Solecki, S. A proof of uniqueness of the Gurariĭ space. *Israel J. Math. 195*, 1 (2013), 449–456.

[50] Kurka, O. The isomorphism class of c_0 is not Borel. *Israel J. Math. 231*, 1 (2019), 243–268.

[51] Leung, D. H., Li, L., Oikhberg, T., and Tursi, M. A. Separable universal Banach lattices. *Israel J. Math. 230*, 1 (2019), 141–152.

[52] Lindenstrauss, J. On extreme points in ℓ_1. *Israel Journal of Mathematics 4*, 1 (1966), 59–61.

[53] Lindenstrauss, J., and Phelps, R. Extreme point properties of convex bodies in reflexive Banach spaces. *Israel Journal of Mathematics 6*, 1 (1968), 39–48.

[54] Lindenstrauss, J., and Tzafriri, L. *Classical Banach spaces. I.* 1977.

[55] Lindenstrauss, J., and Tzafriri, L. *Classical Banach spaces. II.* 1979.

[56] Lotz, H. P. Minimal and reflexive Banach lattices. *Mathematische Annalen 209*, 2 (1974), 117–126.

[57] Louveau, A., and Rosendal, C. Complete analytic equivalence relations. *Trans. Amer. Math. Soc. 357*, 12 (2005), 4839–4866.

[58] Lupini, M. Fraïssé limits in functional analysis. *Adv. Math. 338* (2018), 93–174.

[59] Lusky, W. The Gurarij spaces are unique. *Arch. Math. (Basel) 27*, 6 (1976), 627–635.

[60] Masumoto, S. Jiang-su Algebra as a Fraïssé Limit. *arXiv:1602.00124* (2016).

[61] Meyer-Nieberg, P. *Banach lattices.* Universitext. Springer-Verlag, Berlin, 1991.

[62] Mityagin, B. Equivalence of bases in Hilbert scales. *Studia Math. (in Russian) 37* (1970), 111–137.

[63] Müller, I. Fraïssé structures with universal automorphism groups. *Journal of Algebra 463* (2016), 134–151.

[64] Oikhberg, T., and Tursi, M. A. Order Extreme Points and Solid Convex Hulls. *The Mathematical Legacy of Victor Lomonosov: Operator Theory 2* (2020), 297–316.

[65] Oikhberg, T., and Tursi, M. A. Renorming AM-spaces. *arXiv:2011.04140* (2020).

[66] Pełczyński, A., and Rosenthal, H. Localization techniques in L^p spaces. *Studia Mathematica 52*, 3 (1975), 265–289.

[67] Rudin, W. Functional analysis, mcgrawhill. *Inc, New York* (1991).

[68] Schaefer, H. *Banach lattices and positive operators.* Springer, New York–Heidelberg, 1974.

[69] Schechtman, G. On Pelczynski's paper "Universal bases". *Israel Journal of Mathematics 22*, 3-4 (1975), 181–184.

[70] Tursi, M. A. A Separable Universal Homogeneous Banach Lattice. *arXiv:2008.06658* (2020).

[71] Uspenskij, V. V. On the group of isometries of the urysohn universal metric space. *Commentationes Mathematicae Universitatis Carolinae 31*, 1 (1990), 181–182.

[72] Zielinski, J. The complexity of the homeomorphism relation between compact metric spaces. *Adv. Math. 291* (2016), 635–645.